RATIOS OF AFFINE FUNCTIONS

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A ratio of affine functions is a function which can be expressed as the ratio of a vector valued
affine function and a scalar affine functional. The purpose of this note is to examine properties
of sets which are preserved under images and inverse images of such functions. Specifically, we
show that images and inverse images of convex sets under such functions are convex sets. Also,
images of bounded, convex polytopes under such functions are bounded, convex polytopes. In
addition, we provide sufficient conditions under which the extreme points of images of convex
sets are images of extreme points of the underlying domains. Of course, this result is useful when
one wishes to maximize a convex function over a corresponding set. The above assertions are
well known for affine functions. Applications of the results include a problem that concerns the
control of stochastic eigenvectors of stochastic matrices.

Key words: Affine Functions, Convex Sets, Ratios of Functions.

1. Introduction

Let \( B \) be a convex subset of \( \mathbb{R}^n \) and let \( h: B \to \mathbb{R}^m \). We say that \( h \) is affine if for
each \( x^1, \ldots, x^k \in B \) and nonnegative numbers \( \alpha_1, \ldots, \alpha_k \) with \( \sum_{i=1}^k \alpha_i = 1 \),
\[ h(\sum_{i=1}^k \alpha_i x^i) = \sum_{i=1}^k \alpha_i h(x^i). \]
We say that \( h \) is the ratio of affine functions if there exist functions \( f: B \to \mathbb{R}^m \) and \( g: B \to \mathbb{R} \) which are affine and where \( g(x) > 0 \) for every \( x \in B \) such that
\[ h(x) = \frac{f(x)}{g(x)} \quad \text{for all} \quad x \in B. \quad (1) \]
Of course, such functions are continuous. Also, every affine function is the ratio of
affine functions (by selecting \( g \) in (1) to be a constant).

The purpose of this note is to examine images and inverse images of ratios of
affine functions. In particular, we extend known results for the case where the
underlying function is affine. For example, we show that images and inverse images
of convex sets under ratios of affine functions are convex. Also, we show that extreme
points of images of convex sets under ratios of affine functions are images of extreme
points of the underlying domains. Finally we show that the images of bounded
convex polytopes under such functions are bounded convex polytopes. Of course,
all of the above assertions are known for affine functions.

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We next demonstrate that some known results for affine functions do not have corresponding modifications for ratios of affine functions. For example, images of convex polytopes under affine functions are known to be convex polytopes. However, the corresponding result for ratios of affine functions need not hold. For example, let $B = \{ x \in \mathbb{R}^2 : x_2 = 1 \}$ and let $h : B \to \mathbb{R}^2$ be defined by $h(x) = x/(x_1 + x_2)$ for each $x \in B$. Then $B$ is a convex polytope, but $h(B) = \{ x \in \mathbb{R}^2 : x_1 + x_2 = 1, x_2 > 0 \}$ is not a convex polytope (though it is convex). Also images of bounded sets under affine functions are bounded. However, the corresponding result for ratios of affine functions need not hold. For example, let $B = \{ x \in \mathbb{R} : 0 < x_1 \leq 1 \}$ and let $h : B \to \mathbb{R}$ be defined by $h(x) = 1/x$ for each $x \in B$. Then $B$ is bounded, but $h(B) = \{ x \in \mathbb{R} : x > 1 \}$ is unbounded. We observe that ratios of affine functions are neither necessarily convex functions nor concave functions.

We observe that when the range of the underlying function is the real line, weaker assumptions suffice for the conclusion that the images of convex sets are convex. In particular, continuous functions are known to map connected sets into connected sets. Also, it is well known that convex subsets of $\mathbb{R}^n$ are connected and that the reverse holds in the case of $n = 1$, i.e., connected subsets of $\mathbb{R}$ are precisely the convex sets. It immediately follows that a continuous function mapping a subset of $\mathbb{R}^n$ into $\mathbb{R}$ maps convex sets into convex sets. (Of course, ratios of affine functions are continuous.) The above observation concerning real valued continuous functions does not extend to vector valued function, even in the case where the coordinates of the functions are convex. For example, let

$$B = \{ x \in \mathbb{R}^2 : 0 \leq x_1 \leq 1 \text{ and } 0 \leq x_2 \leq 1 \}$$

and let $h : B \to \mathbb{R}^2$ be defined by $h(x) = (x_1, x_1^2 + x_2)$. Then $h$ is continuous and the two coordinates of $h$ are convex, but

$$h(B) = \{ y \in \mathbb{R}^2 : 0 \leq y_1 \leq 1, y_1^2 \leq y_2 \leq y_1^2 + 1 \}$$

is not convex.

Our interest in ratios of affine functions was started by Hartfiel (1981, 1984) which examines stochastic eigenvectors of sets of stochastic matrices. In particular, he considers function $h$ for which (1) holds with $g(x) = \sum_{i=1}^{n} x_i$. In Section 4 we show how our results can be used to obtain some of the results of Hartfiel (1981, 1984). Also, Brayton, Hoffman and Scott (1977) examined real valued functions on sets of matrices which map convex sets into convex sets. Finally, Karmarkar recently used ratios of affine functions and their properties in the development of his new algorithm for solving linear programs.

Following the introduction and some notational conventions in Section 2, we establish our main results concerning ratios of affine functions in Section 3. Two applications of these results are described in Sections 4 and 5. Finally, extensions are discussed in Section 6.

2. Notational conventions

Let $B \subseteq \mathbb{R}^n$ and let $h : B \to \mathbb{R}^m$. The image of a set $C \subseteq B$ under $h$, denoted $h(C)$,