A QUADRATIC PROGRAMMING ALGORITHM USING CONJUGATE SEARCH DIRECTIONS

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A quadratic programming algorithm is presented, resembling Beale's 1955 quadratic programming algorithm and Wolfe's Reduced Gradient method. It uses conjugate search directions. The algorithm is conceived as being particularly appropriate for problems with a large Hessian matrix. An experimental computer program has been written to validate the concepts, and has performed adequately, although it has not been used on very large problems. An outline of the solution to the quadratic capacity-constrained transportation problem using the above method is also presented.

Key words: Quadratic Programming, Reduced Gradient, Conjugate Directions, Transformation of Search Directions, Transportation Problem

1. Introduction

This paper proposes an algorithm for quadratic programming. It is a reduced gradient method. The reduced gradient of the objective function at successive trial solutions is calculated from the gradient of the objective function. This is the only information used about the objective function. The method of conjugate gradients could then be used to develop a finite algorithm using these gradients. But this would require a complete restart when the reduced problem changes.

It is a feature of this algorithm that no such restart is necessary, provided storage is available to store some search directions along which optimization was successful. When the reduced problem changes linear transformations of stored directions are formed which span a subspace of the space in which we optimize the new reduced problem. Since optimization in that subspace has already been completed we have to generate fewer search directions. This facility generally reduces the total computation time.

If the Hessian of the quadratic objective function is not positive semidefinite the method may yield a local optimum which is not a global optimum.

The present work is related to previous work as follows: It resembles Beale's method for quadratic programming [2] and Wolfe's Reduced Gradient method [7, 8]. Conjugate gradients were proposed by Hestenes and Stiefel [5] for solving large sparse sets of linear equations. Conjugate gradients were adapted by

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Section 2 presents some general discussion and algebra. Section 3 introduces the basic concepts of the method. Section 4 describes the use of conjugate directions. Section 5 presents a list of the quantities to be computed. Section 6 presents a summary of the algorithm. In Section 7 a proof of finiteness of the algorithm is given. Section 8 presents some comments on the computational performance. Section 9 outlines the solution to the quadratic capacity-constrained transportation problem.

2. Discussion

The problem solved can be generally written as follows:

\[
\begin{align*}
\text{minimize} & \quad f = f_0 + C^T x + \frac{1}{2} x^T H x, \\
\text{subject to} & \quad A x = b, \\
& \quad L_j \leq x_j \leq U_j \quad \text{for all } j
\end{align*}
\]  

where \( x \) is a \( n \times 1 \) vector of unknown quantities (includes slacks), \( C \) is a \( n \times 1 \) vector of known constants, \( H \) is a \( n \times n \) symmetric matrix of known constants, \( A \) is a \( m \times n \) matrix of known constants, \( b \) is a \( m \times 1 \) vector of known constants, \( f_0 \) is a known constant, \( L_j \) and \( U_j \) are the lower and upper bounds on the quantities \( x_j \).

Any feasible solution must satisfy the relation

\[
x_B = A_B^{-1} b - A_B^{-1} A_R x_R - A_B^{-1} A_y y
\]  

where

\[
x = \begin{bmatrix} x_B \\ y \\ x_R \end{bmatrix}
\]

\( x_B \) are the basic variables. \( y \) are defined as the independent variables. These are nonbasic variables in the space of which optimization is performed. \( x_R \) are called restricted variables. These are nonbasic variables which are temporarily held at their lower or upper bounds. \( A_B, A_y, \) and \( A_R \) are the columns of matrix \( A \) corresponding to the basic, independent, and restricted variables.

The objective function can be written as a function of the independent variables as follows:

Let the \( n \times n_r \) matrix \( R \) equal

\[
R = \begin{bmatrix} -A_B^{-1} A_y \\ 1 \\ 0 \end{bmatrix}
\]  

\( n_0 \) \( n_n \) \( n_r \)