Minimization of a quasi-concave function over an efficient set

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The nonconvex programming problem of minimizing a quasi-concave function over an efficient (or weakly efficient) set of a multiobjective linear program is studied. A cutting plane algorithm which finds an approximate optimal solution in a finite number of steps is developed. For the particular "all linear" case the algorithm performs better, finding an optimal solution in a finite time, and being more easily implemented.

Key words: Multiobjective programming, efficiency, weak-efficiency, nonlinear programming, nonconvex programming.

1. Introduction


On the other hand the problem of minimizing a numerical linear function $f$ over an efficient set was approached by Philip [16] in 1972, Benson [1], [2], [3], [4] in 1984, 1986, 1991 and Craven [7] in 1991. Also an interesting particular case when $f$ is one of the multiobjectives of a vector maximization problem, was considered by Isermann and Steuer [9] in 1987. The main difficulties of this problem are given by the facts that the efficient set is not convex and is not explicitly described. Some nonlinear cases have been studied by Dauer [8], and Bolintineanu in [5], [6], where the attention was focused on local solutions.

In this paper we consider the difficult problem of globally minimizing a nonconvex function, namely a quasi-concave one, over the efficient set given by a multiobjective linear program hence, a nonconvex set. Finally we will describe an algorithm to obtain an approximate (global) solution in a finite number of steps. The algorithm can be easily adapted when the efficient set is replaced by the weakly-efficient set.

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Also, a special attention is paid for the particular "all linear" case (i.e., the function \( f \) is linear), when the algorithm is more easily implemented than in the general quasi-concave case, and leads to an optimal solution in finite time. In this case the algorithm develops an idea which was briefly outlined by Philip in [16].

2. Preliminaries and notations

We will deal with the optimization problem

\[
(P) \quad \min_{x \in E} f(x)
\]

where \( f : \mathbb{R}^n \rightarrow \mathbb{R} \) is continuous and quasi-concave, \( E \) is the efficient set of the linear vector minimization problem:

\[
(VM) \quad \text{Min}_{x \in S} \quad Cx
\]

with the feasible set \( S \) a polytope (i.e. polyhedral and bounded) given by \( S = \{ x \in \mathbb{R}^n : Ax = b, x \geq 0 \} \), \( A \in \mathbb{R}^{m \times n} \), \( b \in \mathbb{R}^m \) \((m < n)\) and the multiobjectives are given by the matrix \( C \in \mathbb{R}^{r \times n} \). We can assume without loss of generality that the matrix \( A \) has the rank \( m \). (We could also consider the general case when \( S \) is described by linear inequalities and/or equalities, and not all the variables \( x_i \) are sign restricted, but using "slack" variables we can reduce the problem to the previous one as is usually done in Linear Programming.)

By definition \( E = \{ x \in S : (\forall x' \in S) Cx' - Cx \notin \mathbb{R}^r_+ \setminus \{0\} \} \). Note that all the results of this paper hold if we replace \( E \) by the weakly-efficient set \( E_w = \{ x \in S : (\forall x' \in S) Cx' - Cx \notin \text{int} \mathbb{R}^r_+ \setminus \{0\} \} \), and then we have to replace throughout the paper the relation \( "\lambda \in \text{int} \mathbb{R}^r_+" \) by \( "\lambda \in \mathbb{R}^r_+ \setminus \{0\}" \), and in the systems (15), (33), (32), the condition \( "\lambda > 0" \) becomes \( "\lambda^T e = 1 \) and \( \lambda \geq 0" \), where \( e \) is a vector of ones.

Let \( V \) be the set of the vertices of \( S \) and \( V_c = V \cap E \) the set of efficient vertices. Recall that "vertex" is synonym for "extremal point" (used in the convex analysis).

Also a set \( B = \{ j_1, \ldots, j_m \} \subset \{ 1, \ldots, n \} \) is called a basis for \( S \) iff the matrix \( A_B = [A_{j_1}, \ldots, A_{j_m}] \) is invertible \((A_i \) stands for the \( j \)th column of the matrix \( A \)). In this case, defining \( x_B = [x_{j_1}, \ldots, x_{j_m}]^T \), and accordingly \( A_N, x_N \), where \( N = \{ 1, \ldots, n \} \setminus B \), the system \( Ax = A_B x_B + A_N x_N = b \) is equivalent to

\[
x_B + A_B^{-1} A_N x_N = A_B^{-1} b \tag{1}
\]

and the solution corresponding to \( x_N = 0 \), hence \( x_B = A_B^{-1} b \), is called basic solution for \( S \). If a basic solution is also feasible (i.e., all its components are nonnegative) is called basic feasible solution (BFS) of \( S \). A BFS is nondegenerate iff all the basic components are positive \((A_B^{-1} b > 0)\). The polytope \( S \) is nondegenerate iff all the BFS are nondegenerate. A point in \( S \) is a vertex iff is a basic feasible solution. However the corresponding basis is not unique unless \( S \) is nondegenerate.

By an edge of \( S \) we understand the segment in \( \mathbb{R}^n \) determined by two distinct adjacent vertices (i.e. by two distinct BFS which can be obtained one from the other.