ON THE CONTINUITY OF A LAGRANGIAN MULTIPLIER FUNCTION IN INPUT OPTIMIZATION

J. SEMPLE* and S. ZLOBEC

McGill University, Department of Mathematics and Statistics, Burnside Hall, 805 Sherbrooke Street West, Montreal, Quebec, Canada H3A 2K6

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We show that a Lagrangian multiplier function in input optimization is generally discontinuous on regions of stability and then we find conditions for its continuity.

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1. Introduction

Input optimization deals with ‘optimization’ of mathematical programming models by stable perturbations of the parameters (input), see [5, 6, 7]. An important theoretical problem in input optimization is to identify and characterize optimal inputs and the corresponding “optimal realizations” of mathematical models. These are states of the model having the property that every stable perturbation of input results in a worse value of the optimal value function. Optimal inputs were recently characterized in [5] (see also [8]) for convex models in terms of the existence of a Lagrangian multiplier function. This paper addresses itself to the question of continuity of the Lagrangian multiplier function. First we show that the function is generally discontinuous even on regions of stability and then we give conditions for its continuity.

Consider the convex mathematical programming model

\[
(P, \theta) \quad \text{Min}_{(x)} f^0(x, \theta) \\
\text{s.t.} \quad f^k(x, \theta) \leq 0, \quad k \in \mathcal{P} \Delta \{1, \ldots, m\},
\]

where \( \theta = (\theta_i) \in \mathbb{R}^p \) is a data vector, \( x = (x_i) \in \mathbb{R}^n \) is the vector variable, the functions \( f^k(\cdot, \cdot) : \mathbb{R}^n \times \mathbb{R}^p \rightarrow \mathbb{R} \) are continuous and \( f^k(\cdot, \theta) : \mathbb{R}^n \rightarrow \mathbb{R} \) are convex for every \( \theta, k \in \{0\} \cup \mathcal{P} \).

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With every fixed \( \theta \) in \((P, \theta)\) we associate

\[
F(\theta) = \{ x \in \mathbb{R}^n : f^k(x, \theta) \leq 0, k \in \mathcal{P} \}: \text{the feasible set},
\]

\( \hat{x}(\theta) \): an optimal solution,

\( \hat{F}(\theta) \): the set of all optimal solutions,

\( \hat{f}(\theta) \): the optimal value.

Suppose that the model \((P, \theta)\) is running with some fixed data (parameter, input) vector \( \theta^* \). Then, around \( \theta^* \), we can study the behavior of the triple \( \{F(\theta), \hat{F}(\theta), \hat{f}(\theta)\} \), considered as output. The triple changes continuously in the 'regions of stability' recalled from [3] (see also [4]).

1.1. Definition. Model \((P, \theta)\) is stable in a region \( S \subset \mathbb{R}^p \) at \( \theta^* \) if, for some neighbourhood \( N(\theta^*) \) of \( \theta^* \), both

(i) \( \theta \in N(\theta^*) \cap S \Rightarrow \hat{F}(\theta) \neq \emptyset \) and

(ii) \( \theta \in N(\theta^*) \cap S \) and \( \theta \to \theta^* \), imply that the set \( \hat{F}(\theta) \) is bounded and all its accumulation points are in \( \hat{F}(\theta^*) \).

When \( \hat{F}(\theta^*) \neq \emptyset \) and bounded, then a region of stability at \( \theta^* \), that is independent of the objective function, is

\[
M(\theta^*) = \{ \theta \in \mathbb{R}^p : F(\theta^*) \subset F(\theta) \};
\]

see [4]. Another region of stability is stated in terms of the minimal index set of active constraints

\[
\mathcal{P}^-(\theta) = \{ k \in \mathcal{P} : x \in F(\theta) \Rightarrow f^k(x, \theta) = 0 \},
\]

and the corresponding subset of \( \mathbb{R}^n \)

\[
F^-(\theta) = \{ x \in \mathbb{R}^n : f^k(x, \theta) = 0, k \in \mathcal{P}^-(\theta) \}.
\]

This region is

\[
V(\theta^*) = \{ \theta : F^-(\theta^*) \subset F^-(\theta) \text{ and } f^k(x, \theta) \leq 0, \forall x \in F(\theta^*), k \in \mathcal{P}^-(\theta^*), k \notin \mathcal{P}^-(\theta) \};
\]

see [4]. Although the continuity of the output is preserved on both \( M(\theta^*) \) and \( V(\theta^*) \), the behaviour of the Lagrangian multiplier function on these regions is shown below to be essentially different.

For a fixed \( \theta \), consider the restricted Lagrangian

\[
L^< (x, u; \theta) = f^0(x, \theta) + \sum_{k \in \mathcal{P}^<(\theta)} u_k f^k(x, \theta).
\]

Here \( \mathcal{P}^<(\theta) = \mathcal{P} \setminus \mathcal{P}^-(\theta) \). Denote \( q(\theta) = \text{card } \mathcal{P}^<(\theta) \) and let \( \mathbb{R}^q_+ \) denote the nonnegative orthant of \( \mathbb{R}^q \).

1.2. Theorem [5]. Consider the convex programming model \((P, \theta)\) at some arbitrary \( \theta \). Then \( x^*(\theta) \in F^-(\theta) \) is an optimal solution if, and only if, there exists a nonnegative