FINITE-DIMENSIONAL VARIATIONAL INEQUALITY AND NONLINEAR COMPLEMENTARITY PROBLEMS: A SURVEY OF THEORY, ALGORITHMS AND APPLICATIONS

Patrick T. HARKER*
Decision Sciences Department, The Wharton School, University of Pennsylvania, Philadelphia, PA 19104–6366, USA

Jong-Shi PANG**
Department of Mathematical Sciences, The Whiting School of Engineering, The Johns Hopkins University, Baltimore, MD 21218, USA

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Over the past decade, the field of finite-dimensional variational inequality and complementarity problems has seen a rapid development in its theory of existence, uniqueness and sensitivity of solution(s), in the theory of algorithms, and in the application of these techniques to transportation planning, regional science, socio-economic analysis, energy modeling, and game theory. This paper provides a state-of-the-art review of these developments as well as a summary of some open research topics in this growing field.

Key words: Variational inequality, complementarity, fixed points, Walrasian equilibrium, traffic assignment, network equilibrium, spatial price equilibrium, Nash equilibrium.

1. Introduction

The computation of economic and game theoretic equilibria has been of great interest in the academic and professional communities ever since the path-breaking paper by Lemke and Howson [150] and the seminal work by Herbert Scarf [236] in the mid-1960’s and early 1970’s. The initial impetus for research on computing equilibria came from the need to empirically analyze general equilibrium theory and to apply this theory to study problems of taxation, unemployment, etc. In recent years, the growth of experimental economics and the use of sophisticated strategic planning models by industry has revitalized the need for efficient methods to analyze and numerically solve models of economic and game theoretic equilibria.

The initial methods which were used to compute economic equilibria were all based on the ingenious constructive proof by Lemke and Howson of the existence

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of an equilibrium point for a bimatrix game, and have come to be known as fixed-point (or homotopy-based) methods. There were many well-known pioneers in this field: Scarf [235] introduced the notion of primitive sets and described the first algorithm to approximate a fixed point of a continuous mapping; Kuhn [139] and Hansen [89] were responsible for the introduction of simplexes; Eaves [50, 51] introduced piecewise-linear maps into the computational fixed-point literature. The classical reference by Todd [259] and the more recent text by Garcia and Zangwill [81] provide detailed discussions of these techniques. Many applications have been and continue to be made of these methods [90, 159, 173, 238, 239, 265]. While theoretically very powerful, homotopy-based methods have experienced difficulty in solving medium to large-scale equilibrium models. The PIES (Project Independence Evaluation System) energy model [5] which was developed at the U.S. Department of Energy in the late 1970's provided a useful piece of practical evidence demonstrating the inability of the fixed-point methods in handling real-life applications.

The other traditional approach for solving equilibrium models is nonlinear optimization (equilibrium programming in [81]). As discussed in Carey [27], this approach requires very restrictive assumptions on the model in order to work. Thus, solving equilibrium models as optimization problems does not provide a satisfactory alternative to the fixed point/homotopy methods.

In summary, the optimization and fixed point approaches each have their own advantages and disadvantages when applied to solve equilibrium models; separately, they either lack the generality or the computational efficiency which is necessary for solving large-scale equilibrium problems. In recent years, finite-dimensional variational inequality and nonlinear complementarity problems have emerged as very promising candidates for filling the gap created by the optimization and fixed point approaches. It is with this motivation that we undertake the current research.

To a large extent, the PIES model and the associated PIES algorithm have provided the impetus for the growth of the field of finite-dimensional variational inequality and nonlinear complementarity problems. Historically, the variational inequality problem was introduced by Philip Hartman and Guido Stampacchia in the seminal paper [107], and was subsequently expanded by Stampacchia in several classic papers [152, 155, 249]. For the most part, these early studies of the variational inequality problem were set in the context of calculus of variations/optimal control theory and in connection with the solution of boundary value problems posed in the form of partial differential equations; the books by Kinderlehrer and Stampacchia [130] and Baiocchi and Capelo [15] provide a thorough introduction to these applications of variational inequalities in infinite-dimensional metric spaces. The nonlinear complementarity problem, on the other hand, first appeared in Richard Cottle's Ph.D. dissertation [31], which was later published in [32]. The name "complementarity problem" was not used by Cottle in these two references, but was coined in a subsequent paper dealing with the linear case by Cottle, Habetler and Lemke [36]. In the early years of study on the nonlinear complementarity