FURTHER APPLICATIONS OF A SPLITTING ALGORITHM TO DECOMPOSITION IN VARIATIONAL INEQUALITIES AND CONVEX PROGRAMMING

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A classical method for solving the variational inequality problem is the projection algorithm. We show that existing convergence results for this algorithm follow from one given by Gabay for a splitting algorithm for finding a zero of the sum of two maximal monotone operators. Moreover, we extend the projection algorithm to solve any monotone affine variational inequality problem. When applied to linear complementarity problems, we obtain a matrix splitting algorithm that is simple and, for linear/quadratic programs, massively parallelizable. Unlike existing matrix splitting algorithms, this algorithm converges under no additional assumption on the problem. When applied to generalized linear/quadratic programs, we obtain a decomposition method that, unlike existing decomposition methods, can simultaneously dualize the linear constraints and diagonalize the cost function. This method gives rise to highly parallelizable algorithms for solving a problem of deterministic control in discrete time and for computing the orthogonal projection onto the intersection of convex sets.

Key words: Variational inequality, operator splitting, decomposition, linear complementarity, linear/quadratic programming.

1. Introduction

Let $X$ be a nonempty closed convex set in $\mathbb{R}^n$, the $n$-dimensional Euclidean space, and let $f: X \to \mathbb{R}^n$ be a continuous function. Consider the following problem:

\[ \text{VI}(X, f) \quad \text{Find an } x^* \in X \text{ satisfying } \langle f(x^*), x - x^* \rangle \geq 0 \ \forall x \in X. \]

This problem, called the variational inequality problem, has numerous applications to optimization, including the solution of systems of equations, constrained and unconstrained optimization, traffic assignment, and saddlepoint point problems. (See for example [1, 2, 5, 8, 13, 18].)

We make the following blanket assumptions regarding $f$ and $X$:

**Assumption A.** (a) The function $f$ is monotone, i.e. $\langle f(y) - f(x), y - x \rangle \geq 0 \ \forall x \in X, \forall y \in X$.

(b) The problem $\text{VI}(X, f)$ has a solution.

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Let $D$ be an $n \times n$ positive definite matrix. Consider the following algorithm for solving $VI(X, f)$, whereby the original variational inequality is approximated by a sequence of affine variational inequalities:

**Asymmetric Projection (AP) algorithm**

Iter. 0. Start with any $x^0 \in X$.

Iter. $r + 1$. Given an $x^r \in X$, compute a new iterate $x^{r+1} \in X$ satisfying

$$\langle D(x^{r+1} - x^r) + f(x^r), x - x^{r+1} \rangle \geq 0 \quad \forall x \in X. \quad (1.1)$$

(The iteration (1.1) is well defined because $D$ is positive definite [5, Section 3.5], [18, Section 2].) We have called the above algorithm the asymmetric projection (AP) algorithm because if $D$ is symmetric, then it reduces to the well-known projection algorithm [38] (also see [5, 9, 14, 18, 30]),

$$x^{r+1} = \arg\min_{x \in X} \{\|x - x^r + D^{-1}f(x^r)\|_D\}, \quad r = 0, 1, 2, \ldots,$$

where $\| \cdot \|_D$ denotes the norm $\|x\|_D = (x, Dx)^{1/2}$.

It has been shown that if $D$ and $f$ satisfy a certain contraction condition [9, 30], then $\{x^r\}$ generated by the AP iteration (1.1) converges to a solution of $VI(X, f)$. Unfortunately, this condition implies that $f$ is strictly monotone, which excludes from consideration important special cases of $VI(X, f)$ such as linear complementarity problems and linear/quadratic programs. The goal of this paper is two-fold: Firstly, we show that the existing convergence conditions for the AP algorithm follow as a corollary of a general convergence condition given by Gabay [12] for a forward–backward splitting algorithm. This leads to a unified and a much simpler characterization of the convergence conditions. Secondly, we show that the convergence condition for the AP algorithm can be broadened such that it is applicable to all monotone (not necessarily strictly monotone) affine variational inequality problems. In particular, we apply this algorithm to linear complementarity problems (for which $X$ is the non-negative orthant) to obtain a matrix splitting algorithm that is simple and, for linear/quadratic programs, massively parallelizable. Unlike existing matrix splitting algorithms, [20, 22, 23, 28] this algorithm requires no additional assumption (such as symmetry) on the problem data for convergence. We also apply this algorithm to generalized linear/quadratic programming problems to obtain a new decomposition method for solving these problems. This method has the important advantage that it can simultaneously dualize any subset of the constraints and diagonalize the cost function; hence it is highly parallelizable. We describe applications of this method to a problem of deterministic control in discrete time and to computing the orthogonal projection onto the intersection of convex sets.

This paper proceeds as follows: In Section 2 we describe the forward–backward splitting algorithm and a convergence result of Gabay for this algorithm. In Section 3 we show that the AP algorithm is a special case of this splitting algorithm and that Gabay’s result contains as special cases existing convergence results for the AP