LINEAR INEQUALITIES, MATHEMATICAL PROGRAMMING 
AND MATRIX THEORY *

Abraham Berman

Université de Montréal, Montréal 101, Canada

and

Adi Ben-Israel

Technion – Israel Institute of Technology, Haifa, Israel

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A survey is made of solvability theory for systems of complex linear inequalities. This theory is applied to complex mathematical programming and stability and inertia theorems in matrix theory.

Introduction

This paper is a survey of solvability theory for the following systems of complex linear inequalities.

\[ T x = b, \ x \in S . \]  
(Section 1, theorem 1)

\[ T x = b, \ x \in \text{int} \ S . \]  
(Section 3, theorem 3)

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\[ Tx \in \text{int } S_1, \ x \in \text{int } S_2 \]  
(Section 3, theorem 4)

where \( T \in C^{m \times n}, b \in C^m \) and \( S, S_1, S_2 \) are suitable cones.

Theorem 1 is a generalization of the Farkas lemma and of a theorem of Levinson, while theorems 3 and 4 imply generalizations of theorems of the alternative of Gordan and Stiemke respectively.

In section 2, theorem 1 is applied to derive duality theorem of complex linear programming, which generalizes the duality theorem of real linear programming and a duality theorem of Levinson.

In section 4, the solvability theory is applied to matrix spaces with suitable inner products and matrix cones. Theorem 5 is a matrix application of theorems 3 and 4. Other applications are mentioned in the remarks which conclude the paper.

0. Notations and preliminaries

\[ C^n, [R^n] \] the \( n \) dimensional complex [real] vector space
\[ C^{m \times n}, [R^{m \times n}] \] the \( m \times n \) complex [real] matrices
\[ R^n_+ \] the nonnegative orthant in \( R^n \).

For any \( x, y \in C^n \):
\( (x, y) \) the inner product of \( x \) and \( y \)
\( \text{Re } x \) the real part of \( x \).

For any \( A \in C^{m \times n} \):
\( A^\tau \) the transpose of \( A \)
\( A^{\text{H}} \) the conjugate transpose of \( A \)
\( R(A) \) the range of \( A \).
\( N(A) \) the null space of \( A \).

For \( A \in C^{n \times n} \):
\( \text{tr}(A) \) the trace of \( A \)
\( \sigma(A) \) the spectrum of \( A \)
\( A^{-1} \) the inverse of \( A \).

For any \( S_1, S_2 \subset C^n \):
\( S_1 \times S_2 \) the cartesian product of \( S_1 \) and \( S_2 \)
\( \text{int } S_1 \) the interior of \( S_1 \).