ON THE NUMBER OF SOLUTIONS TO A CLASS OF COMPLEMENTARITY PROBLEMS*

M. KOJIMA
Department of Information Sciences, Tokyo Institute of Technology, Tokyo, Japan

R. SAIGAL
Northwestern University, Evanston, IL, U.S.A.

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In this paper we consider the problem of establishing the number of solutions to the complementarity problem. For the case when the Jacobian of the mapping has all principal minors negative, and satisfies a condition at infinity, we prove that the problem has either 0, 1, 2 or 3 solutions. We also show that when the Jacobian has all principal minors positive, and satisfies a condition at infinity, the problem has a unique solution.

Key words: Complementarity Problem, Cones, Topological Degree.

1. Introduction

Let \( R^n \) be the \( n \)-dimensional Euclidean space, \( K \subseteq R^n \) be a closed, convex and polyhedral cone that is pointed (i.e., \( K \cap -K = \{0\} \)) and \( R^n_+ \) be the subset of all non-negative vectors in \( R^n \). Given a mapping \( f : K \rightarrow R^n \), and an \( n \)-vector \( q \) in \( R^n \), in this note we consider the problem of establishing the number of solutions to the problem of finding an \( x \) such that

\[
x \in K, \, f(x) - q \in K^+, \quad (x, f(x) - q) = 0
\]

where \( K^+ \) is the polar cone of \( K \), i.e., \( K^+ = \{y : (x, y) \geq 0 \text{ for all } x \in K\} \). In case \( K = R^n_+ \), this problem is called the non-linear complementarity problem, and has been considered by several authors. A partial list of these include Cottle [1], Karamardian [4], Megiddo and Kojima [11], and Saigal and Simon [16].

Our aim in this paper is to make some statement about the solution set of (1.1) for all \( q \) in \( R^n_+ \). For the special case when \( f \) is affine, and \( K = R^n_+ \), two such results exist, namely those of Kojima and Saigal [8] when the Jacobian of \( f \) has

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all principal minors negative and of Samelson et al. [17] and Murty [12] when the Jacobian of \( f \) has all principal minors positive. For the nonlinear case considered in this paper, we will assume

\[
f(0) = 0 \quad \text{and} \quad f \text{ is continuously differentiable.} \tag{1.2}
\]

And when \( Df(x) \) has all principal minors negative for each \( x \) in \( K = R^n_+ \), we will reproduce the main result of Kojima and Saigal [8] that for any \( q \) \((1.1)\) has 0, 1, 2, or 3 solutions. This will be established with \( f \) satisfying the additional assumption

for any sequence \( \{x_k\}_{k=1}^\infty \) in \( R^n_+ \) such that \( \|x_k\| \to \infty \), there exists a subsequence \( J \) such that either there is a \( j \) with \( f_j(x_k) \to -\infty \) for \( k \) in \( J \); or, there is a \( j \) with \( x^+_j > 0 \) for all \( k \) in \( J \) and \( f_j(x_k) \to \infty \) for \( k \) in \( J \).

In addition, using the method of Kojima and Saigal [7] (see also Mas-Colell [10]) when \( f \) satisfies \((1.2)\) and \( K \) is a polyhedral cone, with an appropriate condition on the Jacobian \( Df(x) \) we will show that for each \( q \) \((1.1)\) has a unique solution. When \( K = R^n_+ \), this condition reduces to the fact that \( Df(x) \) is a P-matrix, (i.e., has all principal minors positive), and \( f \) satisfies some condition at infinity.

The principal tool used in the proof of the above mentioned results is degree theory. We will follow the notation of Ortega and Rheinboldt [14] for this purpose. To facilitate the use of this theory, we now formulate \((1.1)\) as an equation solving problem, suggested by Megiddo and Kojima [11].

Define the projection mapping \( P : R^n \to K \) by

\[
\|P(x) - x\| = \min_{y \in K} \|y - x\|
\]

and the vectors

\[
x^+ = P(x) \in K
\]

and

\[
x^- = x - P(x) \in -K^+.
\]

Note that \( \langle x^-, x^+ \rangle = 0 \).

In case \( K = R^n_+ \), the above operation simplifies to

\[
x^+_i = \max\{0, x_i\},
\]

\[
x^-_i = \min\{0, x_i\}.
\]

Now, define the mapping \( g : R^n \to R^n \) by

\[
g(x) = x^- + f(x^+).
\]

For some \( q \) in \( R^n \), consider the problem of solving the system of equations

\[
g(x) = q. \tag{1.4}
\]