SECOND ORDER SENSITIVITY ANALYSIS AND ASYMPTOTIC THEORY OF PARAMETRIZED NONLINEAR PROGRAMS

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In this paper we study second-order differential properties of an optimal-value function \( \varphi(x) \). It is shown that under certain conditions \( \varphi(x) \) possesses second-order directional derivatives, which can be calculated by solving corresponding quadratic programs. Also upper and lower bounds on these derivatives are introduced under weaker assumptions. In particular we show that the second-order directional derivative is infinite if the corresponding quadratic program is unbounded. Finally sensitivity results are applied to investigate asymptotics of estimators in parametrized nonlinear programs.

Key words: Marginal Function, Optimal Value, Nonlinear Programming, Sensitivity Analysis, Parametric Programming, Asymptotic Distribution.

1. Introduction

In this paper we consider the following mathematical programming problem

\[
(\mathcal{P}_x) \quad \text{minimize} \quad f(x, y) \quad \text{in} \quad y \in \mathbb{R}^n
\]

subject to

\[
g_i(x, y) = 0, \quad i = 1, \ldots, q,
\]

\[
g_i(x, y) \leq 0, \quad i = q + 1, \ldots, p,
\]

where vector \( x \in \mathbb{R}^m \) is viewed as a parameter vector giving perturbations of \( (\mathcal{P}_x) \). For each value of \( x \) the set of feasible solutions is

\[
\mathcal{Y}(x) = \{ y: g_i(x, y) = 0, i = 1, \ldots, q; g_i(x, y) \leq 0, i = q + 1, \ldots, p \}
\]

and the optimal-value (sometimes called marginal) function \( \varphi(x) \) is defined by

\[
\varphi(x) = \inf\{ f(x, y): y \in \mathcal{Y}(x) \},
\]

and \( \varphi(x) = +\infty \) if the set \( \mathcal{Y}(x) \) is empty.

First-order differential properties of \( \varphi(x) \) have been studied by many authors (see [8, 9, 17, 18] and references contained therein). Also continuity and differentiability properties of an optimal solution of \( (\mathcal{P}_x) \) have been discussed extensively in the literature on stability and sensitivity analysis, e.g., [2, 6, 7, 10, 14, 16, 20, 23].
The aim of this paper is twofold. Firstly, we examine second-order differential properties of \( \varphi(x) \) and the corresponding optimal solution. Then we consider a situation where the parameter vector \( x \) is subject to random perturbations about some 'true' mean value \( x_0 \). Our second goal will be to investigate the statistical behaviour of program \( (P_x) \) when the unknown value \( x_0 \) is substituted by an estimate \( \hat{x} \) based on a sample of size \( N \). We assume that \( N^{1/2}(\hat{x} - x_0) \) has asymptotically normal distribution and study the asymptotics of program \( (P_x) \). It may be mentioned that the assumption of asymptotic normality usually can be justified by the Central Limit Theorem if the estimate \( \hat{x} \) is taken to be the arithmetical mean of a sample (see, e.g., [13, Section 2c.5]). Knowing the (asymptotic) distribution of \( \varphi(\hat{x}) \) and of the corresponding optimal solution may be useful, for example, in stochastic programming (see [4] and references therein). Also somewhat close ideas have been applied by the author in [19] to give a unified treatment of the asymptotic distribution theory in the analysis of covariance structures.

The organization of this paper will be as follows. In Section 2 we briefly summarize some mostly known results from sensitivity analysis. The main development is given in Sections 3 and 4 where a second-order sensitivity analysis of \( (P_x) \) is presented. Here our approach is different from previous work in the sense that we do not rely on the Implicit Function Theorem to explain local behaviour of the optimal solution but rather show how \( (P_x) \) can be approximated by the corresponding quadratic program. We shall benefit from this in simplifying the theory and in obtaining stronger results under weaker assumptions. The results of sections 3 and 4 may be considered as an extension and generalization of an investigation initiated by the author in [20]. Finally in section 5 we study statistical implications of sensitivity results from sections 2, 3 and 4. The approach there is rather straightforward and is based on standard techniques of multivariate analysis (cf. [4, 19]).

Throughout the paper, unless stated otherwise, we suppose that the functions \( g_1, \ldots, g_p \) and \( f \) are twice continuously differentiable on \( \mathbb{R}^m \times \mathbb{R}^n \) and that the feasible set \( \mathcal{Y}(x_0) \) corresponding to the point \( x_0 \) is nonempty. We denote by \( M(x) \) the set of those \( y \) which yield the minimum to \( f(x, \cdot) \) over \( \mathcal{Y}(x) \).

2. Basic theory of sensitivity analysis

In this section we summarize some basic sensitivity results for program \( (P_x) \).

**Assumption 1.** There exist a number \( \alpha \) and a compact set \( S \subset \mathbb{R}^n \) such that \( \alpha > \varphi(x_0) \) and

\[
\{ y \in \mathcal{Y}(x): f(x, y) \leq \alpha \} \subset S
\]

for all \( x \) in a neighbourhood of \( x_0 \).

Assumption 1 ensures that for all \( x \) near \( x_0 \) the optimization procedure actually takes place in the compact set \( S \). Consequently the set \( M(x) \) of optimal solutions