\( \mathfrak{C} \)-Extensions of Topological Spaces II

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Abstract

In this paper, we obtain analogues, in the situation of \( \mathfrak{C} \)-extensions, of Magill's theorem on lattices of compactifications. We define an epireflective subcategory of the category \( T_2 \) of all Hausdorff spaces to be admissible (respectively finitely admissible) if for any \( \mathfrak{C} \)-regular space \( X \), every Hausdorff quotient of \( \beta_{\mathfrak{C}}X \) which is Urysohn on \( \beta_{\mathfrak{C}}X - X \) (respectively which is finitary on \( \beta_{\mathfrak{C}}X - X \)) and which is identity on \( X \), has \( \mathfrak{C} \). We notice that there are many proper epireflective subcategories of \( T_2 \) containing all compact spaces and which are admissible; there are many such which are not admissible but finitely admissible. We prove that when \( \mathfrak{C} \) is a finitely admissible epireflective subcategory of \( T_2 \), then the lattices of finitary \( \mathfrak{C} \)-extensions of two spaces \( X \) and \( Y \) are isomorphic if and only if \( \beta_{\mathfrak{C}}X - X \) and \( \beta_{\mathfrak{C}}Y - Y \) are homeomorphic. Further if \( \mathfrak{C} \) is admissible, then the lattices of Urysohn \( \mathfrak{C} \)-extensions of \( X \) and \( Y \) are isomorphic if and only if \( \beta_{\mathfrak{C}}X - X \) and \( \beta_{\mathfrak{C}}Y - Y \) are homeomorphic.

0. Preliminaries: For the sake of completeness, we give below certain fundamental definitions. For further details, see \([1]\) and any elementary book on category theory.

A subcategory \( \mathcal{B} \) of a category \( \mathcal{A} \) is said to be full if for every \( C, D \) in \( \mathcal{B} \), \( \text{Hom}_{\mathcal{A}}(C,D) = \text{Hom}_{\mathcal{B}}(C,D) \). \( \mathcal{B} \) is said to be replete, if every object \( D \) in \( \mathcal{A} \) which is isomorphic to some object \( C \) in \( \mathcal{B} \) is an object of \( \mathcal{B} \). By a subcategory, we mean a full replete subcategory.

A subcategory \( \mathcal{B} \) of a category \( \mathcal{A} \) is said to epireflective, if for every object \( A \) in \( \mathcal{A} \), there exists an object \( \beta A \) in \( \mathcal{B} \) and a unique epimorphism \( \beta \) in \( \text{Hom}(A,\beta A) \), such that for any \( C \) in \( \mathcal{A} \) and any morphism \( f \) in \( \text{Hom}(A,C) \), there exists a unique \( g \) in \( \text{Hom}(\beta A,C) \) such that \( g \circ \beta = f \). We call \( \beta A \) to be the reflection of \( A \) in \( \mathcal{B} \). Also \( \beta \) is called the reflector of \( A \) in \( \mathcal{B} \).

A space \( Y \) is said to be an \( \mathfrak{C} \)-extension of a space \( X \), if \( X \) is dense in \( Y \) and \( Y \) belongs to \( \mathfrak{C} \).
1. **Convention**: \( \mathcal{E} \) is an epireflective subcategory of the category \( T^2 \) of all Hausdorff spaces. Spaces \( X \) considered in this paper are \( \mathcal{E} \)-regular; i.e., they are subspaces of products of spaces in \( \mathcal{E} \). We assume further that \( X \) does not belong to \( \mathcal{E} \). (When \( X \) itself has \( \mathcal{E} \), the situation is trivial and so need not be considered here.) It is also assumed that \( X \) is open in the largest \( \mathcal{E} \)-extension \( \beta_{\mathcal{E}}X \). Further we assume that \( \beta_{\mathcal{E}}X \) is normal.

*Note*: The last two conditions are a bit strong. However when \( \mathcal{E} \) is the category of compact \( T^2 \)-spaces and \( X \) is locally compact, these conditions are satisfied. Certain other situations where these conditions are satisfied have been studied by the author in [6].

2. **Notations**: \( F_\beta(X) \) denotes the family of Hausdorff quotients of \( \beta_{\mathcal{E}}X \), which are identity on \( X \) and which have only a finite number of multiple points.  
\( U_\beta(X) \) denotes the family of all Hausdorff quotients of \( \beta_{\mathcal{E}}X \), which are identity on \( X \) and which are Urysohn on \( \beta_{\mathcal{E}}X - X \). \( \mathcal{E}_\beta(X) \) denotes the family of Hausdorff quotients of \( \beta_{\mathcal{E}}X \) which are identity on \( X \).

*Note*: \( F_\beta(X) \subset U_\beta(X) \subset \mathcal{E}_\beta(X) \).

3. **Definition**: An epireflective subcategory \( \mathcal{E} \) of \( T^2 \) is said to be admissible (respectively finitely admissible), if for any space \( X \), every member of \( U_\beta(X) \) (respectively \( F_\beta(X) \)) has \( \mathcal{E} \).

*Note*: Trivially “admissible” implies “finitely admissible”.

4. **Examples**

(i) Compactness is admissible.

(ii) The property \( \mathcal{C} \) — the closure of any subset of cardinality \( m \) is compact (\( m \) any chosen infinite cardinal) is epireflective and imaginative (i.e., any continuous image inherits the property) and therefore admissible. Clearly when \( m \) is infinite, it is a proper subcategory of \( T^2 \) (cf. [2]).

(iii) Let \( X \) be a topological space. We say that a net \( S \) in \( X \) is weakly open- (or closed-) universal, if given any open set \( A \) in \( X \), \( S \) is either eventually in \( A \) or eventually in \( cA \). The space \( X \) is said to be an \( \alpha' \)-space, if every \( \sigma \)-directed weakly open-universal net in it is convergent; to be an \( \alpha'' \)-space, if every \( \sigma \)-directed universal net in it is convergent. The class of \( \alpha' \)-spaces as well as that of \( \alpha'' \)-spaces is an epireflective, finitely admissible subcategory of \( T^2 \). For further details, see [5].