THE UNIFORM, REGULAR DIFFERENTIAL EQUATIONS OF THE KS TRANSFORMED PERTURBED TWO-BODY PROBLEM*

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(Received 15 November, 1973)

Abstract. The Newtonian differential equations of motion for the two-body problem can be transformed into four, linear, harmonic oscillator equations by simultaneously applying the regularizing time transformation \( \frac{dt}{ds} = r \) and the Kustaanheimo-Stiefel (KS) coordinate transformation. The time transformation changes the independent variable from time to a new variable \( s \), and the KS transformation transforms the position and velocity vectors from Cartesian space into a four-dimensional space. This paper presents the derivation of uniform, regular equations for the perturbed two-body problem in the four-dimensional space. The variation of parameters technique is used to develop expressions for the derivatives of ten elements (which are constants in the unperturbed motion) for the general case that includes both perturbations which can arise from a potential and perturbations which cannot be derived from a potential. These element differential equations are slightly modified by introducing two additional elements for the time to further improve long term stability of numerical integration.

1. Introduction

Stiefel and Scheifele (1971) present a method for linearizing the two-body problem. This method consists of changing the independent variable from time to a new variable, which is proportional to the eccentric anomaly in the elliptic case or its equivalent in the hyperbolic case. The method then changes the coordinates from three-dimensional Cartesian space to a four-dimensional space. The resulting equation for Keplerian motion is a four-dimensional harmonic oscillator.

Stiefel and Scheifele (1971), moreover, showed that, when perturbations are added, the numerical efficiency and accuracy of the new method are considerably greater than for the integration of the Newtonian equations of motion either by direct numerical integration or by the Encke method. In particular, differential equations for ten regular elements, which are constants in the two-body case, can be obtained which are extremely accurate and efficient. By performing a second change in the independent variable to exactly the eccentric anomaly in the elliptical case, an equation in four dimensions could be obtained that is not only linear but has a constant frequency. This last formulation, however, although extremely accurate and efficient, has the disadvantage of being valid only for the perturbed elliptical case; for the near parabolic case it presents numerical difficulties.

Another solution presented by Stiefel and Scheifele (1971) also involves ten elements and is a uniform, regular solution to the perturbed two-body problem. It is regular in the sense that it remains valid for \( r = 0 \), and it is uniform in the sense that it is valid for all values of the energy (that is, the same equations describe the motion whether it is elliptic, parabolic, or hyperbolic). This solution was done for the case of forces arising from a potential only and was done by canonical mechanics. The results of this solution were left in terms of the Hamiltonian and its partial derivatives, not in a form amenable for programming a computer.

The purpose of this paper is twofold. First, it presents the derivation of the uniform, regular element solution for the more general case of perturbations which can arise from a potential and perturbations which cannot be derived from a potential. The derivation is performed by the method of variation of parameters, and explicit expressions are presented for the derivatives of the ten elements of the perturbed motion. Second, because the ten-element solution has mixed secular terms (terms involving the independent variable multiplying a trigonometric function) which tend to degrade the long-term accuracy, the solution is modified to eliminate the mixed secular terms. This modification stems from the introduction of two additional elements according to a similar method given by Burdet (1968).

2. Derivation of Uniform, Regular Set of Elements

In an inertial frame the differential equation for perturbed Keplerian motion may be expressed as:

\[
\ddot{r} + \frac{\mu}{r^3} r = - \left( \frac{\partial V}{\partial r} \right)^T + P = F,
\]

where \( \frac{\partial V}{\partial r} \) is the force arising from the perturbing potential \( V \), \( P \) represents other perturbing forces which may or may not be derivable from a potential, and \( F \) is the total of these perturbations. The independent variable is changed from time \( t \) to \( s \), according to

\[
\frac{dt}{ds} = t' = r,
\]

which implies that

\[
\dot{r} = r'/r
\]

and that

\[
\ddot{r} = r''/r^2 - r'\dot{r}'/r^3.
\]

* Derivative notation is

\( (\cdot)' = d(\cdot)/dt; (\cdot)' = d(\cdot)/ds.\)