On Tensor Rates in Continuum Mechanics\textsuperscript{1)}

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The rates of tensors associated with a moving configuration are shown to be materially objective if they are derived from the rates of the tensor images in a fixed configuration. Higher rates are formulated and a higher 'Jaumann rate' is defined as an alternative to the one discussed recently by Prager. Drucker's postulate is employed to confirm stability criteria for bodies under force-type loading and to derive such criteria for other types of prescribed boundary conditions. Arbitrary moving spatial coordinate systems are finally introduced and tensor rate components established with a view toward greatest possible simplicity.

1. Introduction

Current interest in the establishment of constitutive relations of continua has stimulated several very recent publications dealing with the question of tensor rates. Prager [1]\textsuperscript{2)}, in considering various stress rate definitions in cartesian coordinates, has arrived at a definite preference for the definition of Jaumann [2]. Convected coordinates have been employed, among others, by Hill [3], Sedov [4], and Masur [5]. The use of such coordinate systems leads to relatively simple formulas, but some of the material and spatial relationships become less obvious. General formulations in connection with arbitrary (fixed or moving) coordinate systems have been discussed by Jaunzemis [6], and by Naghdi and Wainwright [7].

Tensor rates are generally acceptable only if they satisfy the condition of material objectivity [8]. Several such rate formulations have been proposed in the past [2, 9, 10, 11]. In the present paper it is shown that these are special cases of a general class of rate definitions which are based on the rates of the tensor images in a fixed configuration. This concept is then extended to the derivation of higher rates. Stability criteria are established next for a broad class of loading conditions. The present section contains expository material (which is largely not new), but only to the extent believed necessary for an understanding of the following sections.

In what follows we consider a body in some reference (or 'material') state and in its instantaneous (or 'spatial') state. An element \(d\overrightarrow{R}\) in the material state is mapped into an element \(d\overrightarrow{r}\) in the spatial state by means of the transformation

\[
d\overrightarrow{r} = \overrightarrow{P} \, d\overrightarrow{R} \quad \quad d\overrightarrow{R} = \overrightarrow{P}^{-1} \, d\overrightarrow{r}
\]

in which, for the ordinary continuum being considered here, the tensor \(\overrightarrow{P}\) and its inverse \(\overrightarrow{P}^{-1}\) are identified by the conditions of integrability

\[
\overrightarrow{P} = \overrightarrow{V} \quad \quad \overrightarrow{P}^{-1} = \overrightarrow{R} \overrightarrow{V}.
\]

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\textsuperscript{2)} Numbers in brackets refer to References, page 200.
In Equations (2), \( \mathbf{R} \) and \( \mathbf{r} \) are the position vectors of the corresponding mass particle in the two configurations; the symbols \( \nabla \) and \( \nabla \) designate the gradient operators with respect to the two configurations, respectively.

Similarly the vector \( d\mathbf{N} \), which has the magnitude of an element of area and whose direction is normal to that element, is mapped into \( d\mathbf{n} \) by means of

\[
d\mathbf{n} = \mathbf{P}^* J d\mathbf{N} \quad J d\mathbf{N} = \mathbf{P}^* d\mathbf{n}
\]

in which \( \mathbf{P}^* \) represents the transpose of \( \mathbf{P} \) and \( J \) is the mapping function which connects an element of volume in the material state with that in the spatial state, that is,

\[
db = J dB.
\]

In general, a material vector \( \mathbf{V} \) may be mapped (or 'convected') into its spatial image \( \mathbf{V} \) and back by means of either Equation (1) or Equation (3). In the former case, \( \mathbf{V} \) and \( \mathbf{V} \) will be called 'contravariant images' of one another, and in the latter case 'covariant images'\(^3\); either mapping may or may not involve \( J \). We note that the mapping process just described involves more than a mere 'shifting' of the vector. It is also noted that the scalar product between two vectors remains invariant if one vector is mapped contravariantly and the other covariantly. If the two states are connected through a rigid body transformation the two mapping processes coalesce.

Extending these concepts to second order tensors, we may map a spatial tensor \( \mathbf{s} \) into its material image by one of the following four mapping processes:

\[
\begin{align*}
J^{-1} \mathbf{T} &= \mathbf{S}' = \mathbf{P} \mathbf{s} \mathbf{P}^* \quad \mathbf{S} = \mathbf{P}^* \mathbf{S}' \mathbf{P}^* \quad (a) \\
"\mathbf{s}" &= \mathbf{P}^* \mathbf{s} \mathbf{P} \quad \mathbf{s} = \mathbf{P}^* "\mathbf{s}" \mathbf{P} \quad (b) \\
'\mathbf{s}' &= \mathbf{P} \mathbf{s} \mathbf{P} \quad \mathbf{s} = \mathbf{P} '\mathbf{s}' \mathbf{P} \quad (c) \\
"'\mathbf{s}'" &= \mathbf{P}^* \mathbf{s} \mathbf{P}^* \quad \mathbf{s} = \mathbf{P}^* "'\mathbf{s}'" \mathbf{P}^* \quad (d)
\end{align*}
\]

Of these, case (a) is contravariant and case (b) covariant while cases (c) and (d) are mixed. Symmetry properties of the tensor are preserved by (a) and (b) and its invariants by (c) and (d). As before, \( J \) may also be employed in the transformation; for example, \( \mathbf{T} \) as defined in Equation (5a) represents the Piola-Kirchhoff tensor of the second kind associated with \( \mathbf{s} \).

The Green tensor \( \mathbf{C} \) and its inverse \( \mathbf{C}^{-1} \) are, respectively, the covariant and contravariant material images of the spatial unit tensor \( \mathbf{I} \); similarly, the Cauchy tensor \( \mathbf{c} \) and its inverse \( \mathbf{c}^{-1} \) represent the spatial images of the material unit tensor \( \mathbf{I} \), or

\[
\mathbf{C} = \mathbf{P}^* \mathbf{P} \quad \mathbf{C}^{-1} = \mathbf{P} \mathbf{P}^* \quad (6)
\]

\[
\mathbf{c} = \mathbf{P}^* \mathbf{P} \quad \mathbf{c}^{-1} = \mathbf{P} \mathbf{P}^* \quad (7)
\]

\(^3\) This terminology has been chosen on the basis that in convected coordinates Equations [1] and [3] (without the use of \( J \)) leave the contravariant and covariant components, respectively, unchanged.