Flow Past a Stretching Plate

By Lawrence J. Crane, School of Mathematics, Trinity College, Dublin, Ireland

Sakiadis (1961) investigated the flow due to a sheet issuing with constant speed from a slit into a fluid at rest. This flow was of Blasius type, in which the boundary layer thickness increased with the distance from the slit. An extension to this problem is that of a stretching sheet whose velocity is proportional to the distance from the slit. This occurs in the drawing of plastic films.

The flow in this case has certain similarities with the Hiemenz (1911) boundary layer flow near a stagnation point in which the main velocity in the outer flow is proportional to the distance from the stagnation point. Just as in the Hiemenz flow the boundary layer thickness is constant and a solution of the boundary layer equations of the form

\[ u = \alpha x f(\beta y) \]

is expected; \((x, y)\) are rectangular Cartesian coordinates with origin at the slit, \(x\) is measured along the sheet in the direction of motion and \((u, v)\) are the corresponding velocity components. One important difference is that since \(u = 0\) at the edge of the boundary layer the outer pressure is constant; this leads to a homogeneous Hiemenz type equation which has an exponential solution in \(f\). The velocity components are

\[ u = \alpha x \exp(-\beta y), \quad v = -\alpha \frac{[1 - \exp(-\beta y)]}{\beta}, \quad (1) \]

where \(\beta = \sqrt{\alpha/\nu}\), \(\nu\) is the coefficient of kinematic viscosity, \(\alpha x\) is the velocity of the sheet and \(\alpha\) is a constant. At the edge of the boundary layer there is a transverse component of velocity \(-\alpha/\beta\).

Heat Conduction in the Linear Stretching Case

The boundary layer equation governing the flow of heat is

\[ u \frac{\partial T}{\partial x} + v \frac{\partial T}{\partial y} = \nu \frac{\partial^2 T}{\partial y^2} \quad (2) \]

where \(T\) is the temperature and \(P\) is the Prandtl number. Let the temperature of the sheet have the constant value \(T_s\) and the surrounding fluid the constant temperature \(T_s\) and put

\[ \overline{T} = \frac{(T - T_s)}{(T_p - T_s)}, \quad \eta = \beta y. \quad (3) \]
Then (2) reduces to
\[ -P (1 - \exp(-\eta)) \frac{\partial \overline{T}}{\partial \eta} = \frac{\partial^2 \overline{T}}{\partial \eta^2}, \] (4)
with boundary conditions:
\[ \overline{T} = 1 \text{ on } y = 0 \text{ and } \overline{T} = 0 \text{ at } y = \infty. \] (5)
Equation (4) has the solution:
\[ \overline{T} = \frac{\int_{\eta}^{\infty} \exp[-P(\eta + e^{-\eta})] d\eta}{\int_{0}^{\infty} \exp[-P(\eta + e^{-\eta})] d\eta} \] (6)
When the Prandtl number is unity the solution is simply
\[ \overline{T} = \frac{e}{e - 1} [1 - \exp(-e^{-\eta})], \] (7)
and when the Prandtl number is large (as is the case for lubricating oils which have a Prandtl number of order $10^3$) expression (6) becomes approximately:
\[ \overline{T} = 1 - \text{erf} \left( \beta y \sqrt{\frac{P}{2}} \right) \]
with an error of order $(P)^{-1/2}$.

**The Skin Friction and Heat Transfer Coefficients**

The non-dimensional shear stress coefficient defined to be
\[ \frac{\mu (\partial u/\partial y)_{y=0}}{\rho \alpha^2 x^2} \]
is equal to $R_\infty^{-1/2}$ where $R_\infty$ is the Reynolds number $\alpha x^2/r$. The total drag on both sides of the sheet up to $x$ is $\rho \alpha^2 x^3 (R_\infty)^{-1/2}$. The heat flux from one side of the plate is
\[ k \left( \frac{\partial T}{\partial y} \right)_{y=0} = \frac{k}{x} (T_y - T_0) (R_\infty)^{1/2} H(P), \]
where $k$ is the coefficient of thermal conductivity of the surrounding medium and
\[ H(P) = \frac{P^p \exp(-P)}{\int_{0}^{\infty} x^{p-1} e^{-x} dx}. \]
$H(P)$ is approximately $P, \sqrt{P/2 \pi}$ when $P$ is small and large respectively, while for $P = 1$ it has the value $(e - 1)^{-1}$. 