ON THE FERMAT–WEBER PROBLEM WITH
CONVEX COST FUNCTIONS

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We treat an extension of the generalized Fermat–Weber problem with convex cost functions. It is shown that the entire sequence of iterates (as opposed to selected subsequences) generated by each of the two proposed algorithms converges to a minimum although the economic function is not strictly convex. The general idea is to associate, with the economic function called $h$, a family of more regular strictly convex functions, the lower envelope of which is the function $h$.

Key words: Location Problem, Unconstrained Nondifferentiable Convex Minimization, Interpolation, Entire Convergence.

1. Introduction

In this paper we treat the Fermat–Weber Problem with convex cost functions i.e.:

$$\min \left\{ \sum_{i=1}^{m} \phi_i(d_i(x)) \mid x \in \mathbb{R}^n \right\}.$$ (1)

The functions $\phi_i$ are convex, differentiable and nondecreasing, $d_i(x) = \|x - a_i\|$ (Euclidean norm) and the $a_i$'s are $m$ given points in $\mathbb{R}^n$, called vertices. This problem has been considered by Katz [9].

It includes as a special case the classical generalized Fermat–Weber Problem

$$\min \left\{ \sum_{i=1}^{m} \omega_i d_i(x) \mid x \in \mathbb{R}^n \right\},$$ (2)

where $\omega_i$ is a positive weight. The latter problem has fascinated a great number of mathematicians since Fermat posed it more than 300 years ago.

Problems (1) and (2) arise frequently in economics. They are found in the problem of locating factories, distribution and communication centers (electricity or telephone lines, gas-pipes, pipe-lines).

Moreover they are used in a crucial manner as subproblems in Branch and Bound methods to solve more general combinatorial problems (several warehouses, several factories). For these multi-source problems see Cooper [1, 2], Francis and White [6], Kuenne and Soland [11].

For Problem (1), Weiszfeld [18] defined an iterative algorithm which con-
verges provided the current point is distinct from a vertex, a point where the function is not differentiable. Weiszfeld's iterative formula has been rediscovered several times by many authors. Reviewing the question, Kuhn [10] points out that the possibility of reaching such a vertex (in a finite number of steps) holds for starting points belonging to a certain denumerable subset of points, in which case the algorithm stops. Jacobsen [7] gave a successor of such a vertex and Cordellier, Fiorot and Jacobsen [3] proved the convergence of the resulting algorithm which is well defined everywhere. A similar approach is given by Ostresh [14]. Later on Cordellier and Fiorot [4] gave three different algorithms, the first of which includes the one defined in [3].

Problem (2) has been treated by Planchart and Hurter [17] in the case when the economic function depends linearly on several metrics, for example the euclidean and rectangular metrics. They used among other things Wendell and Peterson's results related to geometric programming duality [19].

In this paper we extend the first two algorithms defined in [4] to Problem (1). Whereas the function of Problem (2) is strictly convex as soon as the vertices $a_i$ are not collinear (otherwise the solution is trivial), the function of Problem (1) is not necessarily so, whether the vertices are collinear or not.

This function is only convex. However in an appendix we shall define on the one hand a necessary and sufficient condition for this function to be strictly convex and on the other hand a necessary and sufficient condition for this function to have no unique minimum. Moreover a corollary characterizes the subset of minimum points. The proofs of these results are omitted and they can be found in [5].

First in Section 2 we introduce notation, hypotheses and some properties and then we give Algorithm I in Section 3. The iterative function of this algorithm is single valued at all points distinct from a vertex and multivalued at a vertex where it is not continuous. We show the convergence of the entire sequence defined by Algorithm I to a minimum. A result of Huard and a lemma related to convergence of sequences are used.

In Section 4 we define Algorithm II in which the iterative function is everywhere single valued and continuous. In this case the convergence is proved using Zangwill's theorem and the lemma mentioned earlier.

Of course this unconstrained optimization problem with nondifferentiable function can be solved by more general methods such as sub-gradient methods: Lemaréchal [13], Wolfe [20] or by other methods: Poljak [16] using divergent series. These methods generate sequences, every accumulation point of which is a solution of the problem.

On the contrary, the two algorithms proposed here are specific to the problem and as we have said above these algorithms give a sequence of points which converges to a minimum. The approach is totally different. The general idea behind these algorithms is to associate, with the economic function called $h$, a family of more regular strictly convex functions depending on a parameter. The