Invariant Transformations in Non-Steady Gasdynamics and Magneto-Gasdynamics

By Colin Rogers, Department of Mathematics, University of Nottingham, Great Britain

1. Introduction

In a recent paper [1], the author introduced reciprocal relations for one-dimensional and more generally, for \([N + 1]\)-dimensional spherically symmetric non-steady gasdynamics. Further, similar transformations were obtained for certain non-steady magneto-gasdynamics flows.

Here, adjoint relations are introduced and the reciprocal relations of [1] generalised. Subsequently, a general formulation of such invariant transformations based on a matrix description of the governing equations is introduced.

It is seen how invariant transformations may be generated by specialisation of the matrices involved, and examples of such mappings are presented.

2. The Adjoint Relations

Invariant properties in non-steady gasdynamics and magneto-gasdynamics have been developed and utilised considerably in recent years. The work of Nikol'skii [2], Tomilov [3], Rykov [4] and Movsesian [5] is particularly of note.

In this section, invariant transformations of the adjoint type are introduced. Adjoint relations in two-dimensional steady flow were first introduced by Haar [6] when concerned with an adjoint variational problem. These mappings were later recognised by Bateman [7] to be of the Bäcklund type. Here, adjoint-type transformations are defined for \([N + 1]\)-dimensional spherically symmetric non-steady gasdynamics.

The governing equations (in Eularian representation), neglecting heat conduction and radiation, are

\[
x^N \varrho_t + (x^N \varrho u)_x = 0 \quad (N = 0, 1, 2),
\]

\[
\varrho (u_t + u u_x) + \varrho_x = 0,
\]

\[
s_t + u s_x = 0,
\]

together with an equation of state. In the above, \(x\) denotes a linear co-ordinate in the one-dimensional case \(N = 0\), while for \(N = 1\) and \(N = 2\), it designates a radial co-ordinate, plane for \(N = 1\) and spatial for \(N = 2\). Further, \(u(x, t)\) denotes the velocity magnitude and \(\varrho, \varrho, s\) the pressure, density and specific entropy respectively.
Equations (1) and (2) together imply the existence of \( x(x, t), t(x, t) \) defined by the matrix equation
\[
\frac{dx}{dt} = \mathbf{A} \frac{d\vec{x}}{dt},
\]
where
\[
dz = \begin{pmatrix} \frac{dx}{dt} \\ \frac{dt}{dt} \end{pmatrix}, \quad \vec{d}z = \begin{pmatrix} \frac{d\vec{x}}{dt} \\ \frac{d\vec{t}}{dt} \end{pmatrix}, \quad \vec{a}_1 \mathbf{A} = \begin{pmatrix} [x^N q + X_1(x)] \vec{a}_1 & -x^N q u \vec{a}_1 \\ [x^N q u + X_2(x)] & -[\vec{\Pi} + x^N q u^2 + \vec{a}_2] \end{pmatrix},
\]
and
\[
\vec{\Pi}(x, t) = \vec{\Pi}(x_0, t) + \int_{x_0}^x x^N p_x \, dx,
\]
x_0 being a convenient reference co-ordinate. Further, \( \vec{a}_1 \) and \( \vec{a}_2 \) are arbitrary real constants, except that \( \vec{a}_1 \neq 0 \), while \( X_1, X_2 \) are arbitrary functions of the variable \( x \).

It will be assumed that
\[
0 < |[(\vec{\Pi} + \vec{a}_2) x^N q + X_1 (\vec{\Pi} + x^N q u^2 + \vec{a}_2) - X_2 x^N q u] \vec{a}_1^{-1}| < \infty \tag{9}
\]
so that singularities in the transformation (4) (and its inverse) are avoided.

Introducing new velocity \( \vec{u} \) according to
\[
\vec{u} = \vec{a}_1 x^N q u \left[ \vec{\Pi} + x^N q u^2 + \vec{a}_2 \right]^{-1}, \quad \vec{a}_1 \neq 0 \tag{10}
\]
it is seen that
\[
\frac{d\vec{z}}{dt} - \vec{u} \frac{d\vec{t}}{dt} = [x^N q (\vec{\Pi} + \vec{a}_2) (\vec{\Pi} + x^N q u^2 + \vec{a}_2)^{-1} + X_1
\]
\[
- x^N q u X_2 (\vec{\Pi} + x^N q u^2 + \vec{a}_2)^{-1} \] \( dx \),
whence, setting
\[
\vec{\rho} = \vec{a}_3 [x^N q (\vec{\Pi} + \vec{a}_2) (\vec{\Pi} + x^N q u^2 + \vec{a}_2)^{-1} + X_1
\]
\[
- x^N q u X_2 (\vec{\Pi} + x^N q u^2 + \vec{a}_2)^{-1}]^{-1}, \quad \vec{a}_3 \neq 0 \tag{11}
\]
the continuity equation
\[
\vec{\rho}_x + (\vec{\rho} \vec{u})_x = 0 \tag{12}
\]
for one-dimensional non-steady gasdynamic flow in the \( x, \vec{t} \)-space is obtained with new flow variables \( \vec{u}, \vec{\rho} \). Further, (4) yields
\[
\frac{d\vec{z}}{dt} = \mathbf{A}^{-1} \frac{d\vec{z}}{dt},
\]
where
\[
\mathbf{A}^{-1} = \begin{pmatrix} \vec{\Pi} + x^N q u^2 + \vec{a}_2 & -x^N q u \\ x^N q u + X_2 & -[x^N q + X_1] \end{pmatrix},
\]
and
\[
\vec{\lambda} = \vec{a}_1 [x^N q + X_1 (\vec{\Pi} + x^N q u^2 + \vec{a}_2) - X_2 x^N q u]^{-1}.
\]
It is noted that \( \vec{\lambda} \neq 0 \) by virtue of condition (9). Hence,
\[
\frac{\partial}{\partial \vec{t}} + \vec{u} \frac{\partial}{\partial \vec{x}} = -\vec{a}_1 (\vec{\Pi} + x^N q u^2 + \vec{a}_2)^{-1} \frac{\partial}{\partial \vec{t}} \tag{13}
\]