On the Generalization of the Cole–Hopf Transformation

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The Cole–Hopf transformation was originally introduced in order to reduce the Burgers equation to the linear heat equation [1, 2]. Several previous studies have dealt with the generalization of this application, the most striking being possibly those of references [3, 4] and [5].

In this note we propose combining the generalization procedures of papers [4] and [5] to yield a large set of equations containing nonlinearity, diffusion and dispersion.

Let \( u(x, t) \) be the dependent variable of the desired nonlinear evolution equation and \( v(x, t) \) the corresponding variable of the associated linear equation. Combining the ideas of [3], [4] and [5], we introduce the ansatz

\[
\begin{align*}
v_x &= F(u, x, t)v \\
v_t &= G(u, x, t)v.
\end{align*}
\]

Here \( F(u, x, t) \) is a function of \( u, x \) and \( t \), and \( G \) is a functional or operator or function of \( u, x \) and \( t \). The compatibility condition for Eqns. (1) and (2) is

\[
\frac{\partial F}{\partial u} u_t + \frac{\partial F}{\partial t} = \frac{\partial G}{\partial u} u_x + \frac{\partial G}{\partial x} + [G, F].
\]

The successive derivations of \( v \) with respect to \( x \) can be expressed as follows:

\[
\begin{align*}
v_x &= Fv \\v_{xx} &= (F^2 + F_x)v \\
&
\end{align*}
\]

Let us take any linear partial differential equation of the first order in time and of arbitrary order in space coordinate:

\[
v_t = \sum_{n=1}^{N} \varphi_n(x, t) \frac{\partial^n}{\partial x^n} v.
\]

Expressing \( v_t \) and \( v_{(n)} \) with the help of Eqns. (1), (2) and (4), we obtain

\[
G = \sum_{n=1}^{N} \alpha_n(x, t)G_n,
\]

and the compatibility condition (3), in which the function \( G \) is specified by Eqn. (6) in terms of \( F \), leads to the desired nonlinear equation for \( u \). For every choice of Eqn. (5)

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and every $F$ we get a specified nonlinear equation whose solution can be expressed through Eqn. (1) in terms of the solutions of Eqn. (5).

A further generalization is possible by taking $u$ and $v$ as $m$ dimensional vectors, and $F$ and $G$ as $m \times m$ matrix functions. Equations (1) and (2) thus become

\[
\frac{\partial}{\partial x} v_\alpha = \sum_\beta F_{\alpha \beta}(u_\beta, x, t) v_\beta \tag{7}
\]

\[
\frac{\partial}{\partial t} v_\alpha = \sum_\beta G_{\alpha \beta}(u_\beta, x, t) v_\beta \tag{8}
\]

The compatibility condition in this case reads

\[
\left[ \sum_\alpha \frac{\partial F}{\partial x} \frac{\partial u_\alpha}{\partial t} + \frac{\partial F}{\partial t} \right] v = \left\{ \sum_\alpha \frac{\partial G}{\partial x} \frac{\partial u_\alpha}{\partial x} + \frac{\partial G}{\partial x} + [G, F] \right\} v. \tag{9}
\]

If we limit discussion to $F$ and $G$ diagonal, the commutator on the right-hand side vanishes and Eqn. (9) constitutes a set of $m$ scalar equations for $m$ unknown functions $u_\alpha$. The solutions for $u_\alpha$ can again be expressed through Eqns. (7) and (8) in terms of the solutions of the linear system

\[
\frac{\partial}{\partial t} v_\alpha = \sum_{j=1}^N \varphi_{\alpha j} \frac{\partial^n}{\partial x^n} v_\alpha. \tag{10}
\]

Similarly to Eqn. (4) we have

\[
G_1 = F, \quad G_n = G_{n-1}G_1 + \frac{\partial}{\partial x} G_{n-1}. \tag{11}
\]

Examples

Using Eqns. (5) and (6), we recover all evolution equations as discussed in [4, 5]. However, more interesting is the generalized transform (7) and (8), which allows one to combine evolution equations of different types. For example, in the case of two variables $u_1, u_2$ we can construct a system

\[
\begin{align*}
 u_{11}u_2 + u_1u_{21} &= \varphi_1(u_{11}, u_2 + u_{12}) + \varphi_2(u_{11}u_2 + 2u_{11}u_{22} + u_1u_{22} + 2u_1u_2u_{11} + 2u_1^2u_2) \\
 u_{11} + \alpha u_{21} &= \psi_1(u_{11} + \alpha u_{21}) + \psi_2(u_{11} + \alpha u_{21} + 2u_1u_{11} + 2\alpha u_1u_{21} + 2\alpha u_{11}u_2 + 2\alpha^2u_2u_{22})
\end{align*}
\]

where $\alpha$ is a constant, and $\psi_1, \psi_2, \varphi_1, \varphi_2$ are arbitrary functions of $x$ and $t$.

References