Relaxation Oscillations of a Josephson Contact

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The voltage-current characteristic of a Josephson junction has been investigated in the adiabatic approximation by numerical methods for solving differential equations [1, 2]. The voltage has been found to be a multivalued function of the current; this gave rise to a hysteresis at finite voltage, at least in a certain range of junction parameters.

The numerical methods used are known to become unstable if the coefficient of the highest time derivative (proportional to the capacitance C) tends to zero. Indeed, besides the inverse Josephson frequency, this introduces a second time unit, the RC relaxation time, which becomes vanishingly small so that the differential equation becomes stiff [3] in this limit.

The limit case of vanishing capacitance will be discussed here by the method of relaxation oscillations, and results for small capacitances will be investigated analytically for the regular case. If the relaxation oscillations are discontinuous, singular perturbation theory has to be used [4, 5].

The Josephson equation in the lowest adiabatic approximation [6, 7] is

$$\frac{Ch\dot{\phi}}{2e} + I_0 \left( \frac{h\dot{\phi}}{2e} \right) - I_1 \left( \frac{h\dot{\phi}}{2e} \right) \cos \varphi + J_1 \left( \frac{h\dot{\phi}}{2e} \right) \sin \varphi = I$$

(1)

where the tunnel functions $I_0, I_1, J_2$ are known as a function of temperature $T$ and energy gaps $\Delta(T)$ for symmetric [8, 9] and asymmetric [9] junctions. In particular, $I_0$ and $I_1$ have a discontinuity at the gap voltage $V_g$, whereas $J_1$ has a logarithmic singularity at $V_g$.

It is convenient to introduce the dimensionless units $t^{(1)} = G_NT/e$ for all currents, $t^{(1)} = h/2T$ for the time and the dimensionless tunnel functions

$$\begin{bmatrix} I_0^{(1)}(V) \\ I_1^{(1)}(V) \\ J_1^{(1)}(V) \end{bmatrix} = \begin{bmatrix} I_0(V) \\ I_1(V) \\ J_1(V) \end{bmatrix}$$

(2)

where $V = eV/T$, and $G_N = 1/R$ is the normal conductance.

Therefore (1) transforms into

$$\beta \varphi'' + i_0(\varphi') - i_1(\varphi') \cos \varphi + j_1(\varphi') \sin \varphi = \alpha$$

(3)

in terms of the reduced bias current $\alpha = eI/G_NT$, time $\tau = t^{(1)}(d/d\tau = \cdot)$ and $\beta = 2TC/hG_N$.

If the capacitance parameter $\beta (\geq 0)$ tends to zero from positive values, i.e., $\beta \rightarrow 0^+$, the junction undergoes relaxation oscillations given by Eqn. (3) without the second-time derivative term, essentially

$$i_0(\varphi') - i_1(\varphi') \cos \varphi + j_1(\varphi') \sin \varphi = \alpha.$$

(4)

By eliminating $\varphi$ in (4) the phase trajectories $\varphi(\varphi')$ are found to be

$$\varphi(\varphi') = \arctg \frac{i_1(\varphi')}{j_1(\varphi')} + \arcsin \frac{\alpha - i_0(\varphi')}{\sqrt{i_0^2(\varphi') + j_1^2(\varphi')}}$$

(5)
which behaves near $\varphi' \approx 2D$ with $D = \Delta(T)/T$ as

$$\varphi(\varphi') \approx \frac{\alpha - (i_0 - i_1)^*}{j_1}$$

if $\varphi \approx 0$,

and as

$$\varphi(\varphi') \approx \pi + \frac{i_0^* + i_1^* - \alpha}{j_1}$$

if $\varphi \approx \pi$,

since $j_1(\varphi') \approx D/2 \cos \pi/2 \ln 16D/|\varphi' - 2D|$ becomes large; $\pm$ refer to the limiting values of $i_{0,1}$ for $\varphi' = 2D \pm 0$. In particular, since $i_0^* - i_1^* = i_0^* - i_1^* = (\pi/2)D/2$ we find

$$i_0^* - i_1^* = i_0^* - i_1^*;$$

hence $\varphi(\varphi')$ always has a cusp near $\varphi \approx 0$, with $\varphi \lesssim 0$ if $\alpha \lesssim (i_0 - i_1)^*$, (see Figure 1, and inset near $\varphi = 0$). Since $i_0^* + i_1^* > i_0^* + i_1^*$, $\varphi(\varphi') - \pi$ has a different sign if $i_0^* + i_1^* < \alpha < i_0^* + i_1^*$, giving rise to an "anticusp" (cases II and III near $\varphi = \pi$); if $\alpha < i_0^* + i_1^*$ or $\alpha > i_0^* + i_1^*$, the phase trajectories near $\varphi = \pi$ have a cusp in either direction (cases I or IV near $\varphi = \pi$). At $\varphi' \approx -2D$ the phase trajectories have cusps at both ends, i.e., in $\varphi = 0$ and $\pi$.

Besides these logarithmic singularities near $\varphi = 0$ and $\pi$, the phase trajectories exhibit other singular points for $d\alpha/d\varphi = 0, d\alpha/d\varphi' = 0$, that is, for

$$\alpha = i_0 \pm \sqrt{i_0^2 + i_1^2}$$

$$0 = i_0' \pm \frac{i_1'j_1' + j_1i_1'}{\sqrt{i_0^2 + j_1^2}}$$

which can also be found as extrema of $\alpha(\varphi, \varphi')$ with $\varphi = -\arctg j_1/i_1$ inserted. This yields the curves $\alpha_{\text{disc}}$ of Figure 1. The remaining curve on Figure 1 gives the maximal Josephson current $j_1(0)$ as a function of $D$ (instead of temperature $T$). The $\beta \to 0^+$ characteristics will be discussed for the following values $D = 0, 1, 2, \infty$, corresponding to the reduced temperatures $T/T_c = 1, 0.9054, 0.7165, 0$.

**Figure 1**

Reduced currents as a function of $D = \Delta(T)/T$, separating ranges with special features of the phase trajectories, e.g., the singularities at $\varphi = 0$ or $2D$ labelled I to IV on the inset. The topology of the phase trajectories changes at $\alpha_{\text{disc}}$, and for $\alpha > j_1(0)$ the stationary solution becomes rotational.