Balanced 0, ±1-matrices, bicoloring and total dual integrality 1

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Abstract

A 0, ±1-matrix A is balanced if, in every submatrix with two nonzero entries per row and column, the sum of the entries is a multiple of four. This definition was introduced by Truemper (1978) and generalizes the notion of a balanced 0,1-matrix introduced by Berge (1970). In this paper, we extend a bicoloring theorem of Berge (1970) and total dual integrality results of Fulkerson, Hoffman and Oppenheim (1974) to balanced 0,±1-matrices.

Keywords: Combinatorial optimization; Integrality of polyhedra; Generalized set packing; Covering

1. Introduction

A 0,1-matrix is balanced if it does not contain a square submatrix of odd order with two ones per row and column. This notion was introduced by Berge [1]. Berge also [2] showed that if A is a balanced 0,1-matrix, then the set packing polytope \( \{ x: Ax \leq 1, 0 \leq x \leq 1 \} \) and the set covering polytope \( \{ x: Ax \geq 1, 0 \leq x \leq 1 \} \) are integral polytopes, i.e., they have only integral vertices.

Definition 1.1. A system of linear constraints is totally dual integral (TDI) if, for each integral objective function vector c, the dual linear program has an integral optimal solution (if an optimal solution exists).

Edmonds and Giles [11] proved that if a linear system is TDI and has an integral right-hand side, then the corresponding polyhedron is an integral polyhedron. Fulkerson,
Hoffman and Oppenheim [12] showed that the linear systems defining the set packing and set covering polytopes are totally dual integral when $A$ is a balanced 0, 1-matrix. So the Edmonds–Giles theorem and the results of Fulkerson et al. imply the results of Berge mentioned above.

A 0, 1-matrix $A$ can be represented by a hypergraph (the columns of $A$ represent nodes and the rows represent edges). Then the definition of balanced 0, 1-matrices is a natural extension of the property that bipartite graphs do not contain odd cycles. This is the motivation that led Berge to introduce the notion of balancedness. For example, balanced hypergraphs can be characterized elegantly by a bicoloring of the nodes (which specializes to the node bipartition in the case of bipartite graphs), see [1], and König’s theorem on bipartite matchings extends to balanced hypergraphs, see [6]. Further results on balanced 0, 1-matrices can be found in [3–5].

A 0, $\pm$1-matrix $A$ is balanced if, in every submatrix with two nonzero entries per row and column, the sum of the entries is a multiple of four. This definition is due to Truemper [18]. The class of balanced 0, $\pm$1-matrices includes balanced 0, 1-matrices and totally unimodular 0, $\pm$1-matrices. (The relation with total unimodularity is made clear by Camion’s theorem [7] which states that a 0, $\pm$1-matrix is totally unimodular if and only if, in every submatrix with an even number of nonzero entries per row and column, the sum of the entries is a multiple of four.) Truemper [18] (see also [19]) gave an important characterization, in terms of excluded submatrices, of the 0, 1-matrices that can be signed into balanced 0, $\pm$1-matrices. The signing being unique up to multiplication of rows and columns by $-1$, Truemper’s theorem provides an excluded submatrix characterization of balanced 0, $\pm$1-matrices. A decomposition theorem for this class of matrices is given in [9].

Given a 0, $\pm$1-matrix $A$, let $n(A)$ denote the column vector whose $i$th component is the number of $-1$’s in the $i$th row of matrix $A$. The generalized set covering polytope is

$$P(A) = \{ x \in \mathbb{R}^p : Ax \geq 1 - n(A), 0 \leq x \leq 1 \}.$$  

The generalized set packing (partitioning) polytope is obtained by replacing $Ax \geq 1 - n(A)$ by $Ax \leq 1 - n(A)$ ($Ax = 1 - n(A)$) in the above definition.

Through a classical transformation, the inference problem in propositional logic can be transformed into a generalized set covering problem

$$\min \{ cx : x \in P(A), x \in \{0,1\}^p \}.$$  

When the matrix $A$ is balanced, we prove in [8] that the polytope $P(A)$ is integral and therefore that the resulting logical inference problem can be solved by linear programming.

In this paper, we extend Berge’s bicoloring theorem and the total dual integrality results of Fulkerson et al. to balanced 0, $\pm$1-matrices.