Applying Lehman's theorems to packing problems

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Abstract

A 0-1 matrix $A$ is ideal if the polyhedron $Q(A) = \text{conv}\{x \in \mathbb{Q}^V: A \cdot x \geq 1, x \geq 0\}$ ($V$ denotes the column index set of $A$) is integral. Similarly a matrix is perfect if $P(A) = \text{conv}\{x \in \mathbb{Q}^V: A \cdot x \leq 1, x \geq 0\}$ is integral. Little is known about the relationship between these two classes of matrices. We consider a transformation between the two classes which enables us to apply Lehman's modified theorem about deletion-minimal nonideal matrices to obtain new results about packing polyhedra. This results in a polyhedral description for the stable set polytopes of near-bipartite graphs (the deletion of any neighbourhood produces a bipartite graph). Note that this class includes the complements of line graphs. To date, this is the only natural class, besides the perfect graphs, for which such a description is known for the graphs and their complements. Some remarks are also made on possible approaches to describing the stable set polyhedra of quasi-line graphs, and more generally claw-free graphs. These results also yield a new class of t-perfect graphs.

Keywords: Ideal matrix; Blocking matrix; t-perfect; Stable set polytope; Total dual integrality

1. Introduction

1.1. Ideal matrices

A 0-1 matrix $A$ is ideal (or has the max-flow min-cut property) if

$$Q(A) = \{ x \in \mathbb{Q}^V: x \geq 0, A \cdot x \geq 1 \}$$

(1)

is an integral polyhedron (where $V$ denotes the index set of columns). It is clear that if $A$ is ideal, then so is the matrix obtained by removing any dominating rows of $A$. Hence
we restrict our attention to matrices without dominating rows. The blocking matrix of $A$, denoted by $b(A)$, is the matrix whose rows consist of all minimal 0–1 vectors in $Q(A)$. It follows [7] that $b(b(A)) = A$.

Evidently, $Q(A)$ is full dimensional. Furthermore, if $A$ has no zero column, then the only nonnegative solution $x$ to $A \cdot x = 0$ is the zero vector. Thus $Q(A)$ is pointed if $A$ has no zero column.

For $c \in Q^+_A$, we denote by $v^*_A(c)$ (or simply $v^*(c)$) the value $\min \{c \cdot x : x \in Q(b(A))\}$ and $\nu^*_A(c)$ the same minimum over the 0–1 vectors in $Q(b(A))$. We define $\tau^*_A(c) = v^*_A(c)$ and similarly for $\tau_A(c)$. Thus $A$ is ideal if and only if $\tau^*_A(c) = \tau_A(c)$ for each $c \in Q^+_V$. For $c$ all ones, we denote these parameters by $\nu^*_A$, $\tau^*_A$ (or simply $\nu$, $\tau$) etc., i.e., the minimum number of ones in a row of $A$ and $b(A)$ respectively. We also denote by $r^*_A$ (or simply $r$) the integer $\max\{0, \nu - |V|\}$.

For node $v \in V$, the contraction of $v$, denoted by $A/v$, is the matrix obtained by removing $v$’s column and deleting any dominating rows. This corresponds to restricting to the face of $Q(A)$ obtained by setting $x_v = 0$ i.e., $Q(A/v) = Q(A) \cap \{x : x_v = 0\}$. The matrix obtained by the deletion of $v$, denoted by $A\setminus v$ is obtained by deleting $v$’s column and keeping only those rows which had a 0 in $v$’s position. This corresponds to restricting to the face of $Q(A)$ obtained by setting $x_v = 1$ i.e., $Q(A\setminus v) = Q(A) \cap \{x : x_v = 1\}$. It is straightforward to check that for $u, v \in V$, $A/v\setminus u = A\setminus u/v$. These two operations are also closed under forming blockers:

**Proposition 1** (Seymour [26]). For a matrix $A$ and $v \in V$, $b(A/v) = b(A)\setminus v$ and $b(A\setminus v) = b(A)/v$.

A minor of $A$ is any matrix of the form $A/T$ for disjoint $S, T \subseteq V$. A $c$-minor (respectively $d$-minor) is such a matrix with $S = \emptyset$ (respectively $T = \emptyset$).

**Proposition 2** (Seymour [26]). If $A$ is ideal, then so too is any minor of $A$.

We also have the following result which resembles the Perfect Graph Theorem (see [17]).

**Theorem 3** (Lehman). A matrix $A$ is ideal if and only if $b(A)$ is ideal.

### 1.2. Minimally nonideal matrices

A minimally nonideal matrix is a matrix which is not ideal such that each of its proper minors is ideal. Note that Theorem 3 implies that a matrix is minimally nonideal if and only if its blocking matrix is minimally nonideal. Since ideal matrices are closed under taking minors, there is a forbidden minor characterization of the class of all such matrices. This is equivalent to characterizing the minimally nonideal matrices. There are three known infinite classes of minimally nonideal matrices. The first two are $C_n$ and