Some Bounds for Quantities associated with Two-dimensional Anisotropic Media

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1. Introduction

In previous work [1, 2] bounds were obtained for the torsional rigidity of certain anisotropic (orthotropic) elastic cylinders in terms of the torsional rigidity of geometrically identical isotropic cylinders and in terms of the relevant physical constants together with readily accessible quantities descriptive of the cross-section geometry. The aim of the present note is to obtain some analogous results for capacitance where the dielectric is anisotropic, and for the lowest eigenvalue in the case of the two dimensional anisotropic wave equation – as exemplified by transverse vibrations of an anisotropically stretched membrane.

2. Bounds for Capacitance

Let the cross-section $D$ of a right cylindrical capacitor be bounded externally and internally by curves of sufficient smoothness which we denote by $S$. We take rectangular cartesian coordinates $x, y$ in the plane $D$, and we suppose that the capacitor is occupied by homogeneous anisotropic dielectric with constants $\mu_1, \mu_2$ corresponding to these directions respectively. The potential $\phi$ satisfies the boundary value problem:

$$
\begin{align*}
\mu_1 & \phi_{xx} + \mu_2 \phi_{yy} = 0 \quad \text{in } D, \\
\mu_1 \phi_x n_x + \mu_2 \phi_y n_y &= 0 \quad \text{on } S_a, \\
\phi &= 1 \quad \text{on } S_1, \\
\phi &= 0 \quad \text{on } S_2,
\end{align*}
$$

where $S_1, S_2, S_a$ are those parts of the boundary $S$ consisting of conducting material at unit potential, conducting material at zero potential, and insulator material respectively, and where the usual notation is used. We may introduce the auxiliary function $\psi$ such that $\mu_1 \phi_x = \psi_y, \mu_2 \phi_y = -\psi_x$, where $\psi$ satisfies the same differential equation as $\phi$, and is constant along every arc of $S_a$. The capacity $C$, per unit length, may be variously expressed in the equivalent forms:

$$
4 \pi C = \iint_{S_1} (\mu_1 \phi_x n_x + \mu_2 \phi_y n_y) \, dS = \iint_{D} (\mu_1 \phi^2_x + \mu_2 \phi^2_y) \, dA = \frac{\int_{S_1} (\mu_1 \psi^2_x + \mu_2 \psi^2_y) \, dA}{(\mu_1 \mu_2)} = \psi_{|S_1},
$$

the usual notation being used, the latter quantity denoting the increment in going around $S_1$ in the usual sense. It may be characterized in variational terms by standard methods [3]:

$$
\iint_{D} (\mu_1 \phi_x^2 + \mu_2 \phi_y^2) \, dA > 4 \pi C > \frac{\mu_1 \mu_2 \{ \psi_{|S_1} \}^2}{\int_{D} (\mu_1 \psi_x^2 + \mu_2 \psi_y^2) \, dA}
$$

where $\hat{\phi}$ is arbitrary to within piecewise continuous differentiability and to within the satisfaction of the conditions on $S_1$ and $S_2$, and where $\hat{\psi}$ is arbitrary to within continuous differentiability and to within the requirement that it be constant on every arc of $S_a$. The upper and lower bounds are attained in (2.3) if and only if $\hat{\phi} = \phi$, and if and only if $\hat{\psi}$ coincides with $\psi$ (or is a constant multiple thereof) respectively. We denote $\phi, \psi, C$ by $\hat{\phi}, \hat{\psi}, \hat{C}$ respectively in the case $\mu_1 = \mu_2 = 1$. 
It is easily established that \( \mu_1 \geq C/C_0 \geq \mu_2 \) where \( \mu_1 \geq \mu_2 \). This may be done by putting \( \bar{\phi} = \phi \) and \( \bar{\psi} = \psi \) in (2.3) and using (2.2). The upper and lower bounds are best possible of their type in general. The upper bound is attained by a rectangular capacitor which has insulator boundaries parallel to the \( x \) axis and which has conducting boundaries (at different potentials) parallel to the \( y \) axis. The upper bound is also attained for a capacitor consisting of two concentric similarly oriented rectangular conductors (at different potentials) with sides parallel to the coordinate axes, the distance between the plates and the distance in the \( x \) direction between points on the outer plate tending to zero in comparison with the length in the \( y \) direction. The lower bound is attained for the same capacitors with the roles of \( x \) and \( y \) interchanged.

If the domain \( D \) is \( n \)-fold symmetric \((n \geq 3)\) and if the boundary conditions on \( \phi \) are consistent with this symmetry it will be recalled [1] that

\[
\int_{\bar{D}} \bar{\phi}_x^2 \, dA = \int_{\bar{D}} \bar{\psi}_y^2 \, dA \quad \text{and} \quad \int_{\bar{D}} \bar{\psi}_x \, dA = \int_{\bar{D}} \bar{\phi}_y \, dA.
\]

Putting \( \bar{\phi} = \phi \) and \( \bar{\psi} = \psi \) in (2.3), using the foregoing results and (2.2), we obtain

\[
\frac{(\mu_1 + \mu_2)}{2} C \geq \frac{2 \mu_1 \mu_2}{(\mu_1 + \mu_2)} \frac{C}{C_0} \geq \frac{C}{C_0} \geq \frac{2 \mu_1 \mu_2}{(\mu_1 + \mu_2)}.
\]

(2.4)

for \( n \)-fold-symmetric cross-sections \((n \geq 3)\), the boundary values taken by the potential being consistent with this symmetry. This result is formally identical with that obtained for the torsional rigidity of orthotropic cylinders with such cross-sections [1]. The upper bound is best possible of its type for such sections being attained for a capacitor with two conducting boundaries (at different potentials) consisting of similarly oriented concentric \( n \)-agons the distance between them tending to zero in comparison with the perimeter length. This may be proved as follows. Denoting side lengths by \( l \) and the distance between plates by \( e \) for a parallel plate condenser the normal to whose plates makes an angle \( \theta + 2 r \pi/n \) with the \( x \) axis, the capacity (per unit length) \( C_r \rightarrow \frac{\mu_1 \sin^2(\theta + 2 r \pi/n) + \mu_2 \sin^2(\theta + 2 r \pi/n)}{l/4 \pi e} \) as \( e/l \rightarrow 0 \). The capacitor mentioned earlier may be considered as an aggregate of \( n \) of these \((1 < r < n)\) joined in parallel, the capacities, of course, being additive, whereupon we obtain \( C = \sum C_r = n l (\mu_1 + \mu_2)/8 \pi e \), on remembering that \( \sum \cos^2(\theta + 2 r \pi/n) = \) \( \sum \sin^2(\theta + 2 r \pi/n) = n/2 \) for \( n \geq 3 \). The attainment of the upper bound is therefore established. It is also attained for a capacitor consisting of two concentric circular conducting boundaries the distance between the circles tending to zero compared to the perimeter. Because of their similarity, the above mentioned sections for which the upper bound is attained invite comparison with some of those sections for which the lower bound is attained in the case of torsional rigidity of orthotropic cylinders [1]. This lower bound is attained for concentric circular cross-sections [1], and for a section bounded by concentric similarly oriented squares, with sides parallel to the axes of orthotropy, as the thickness tends to zero in comparison with the perimeter length. This latter section was not mentioned in [1] but the attainment of the lower bound is easily established in that case. The author is unable to find a case where the lower bound (2.4) is attained within the context and definition of capacity envisaged at the outset. It is of interest, however, to point out that it is attained for a multiple condenser consisting of four equal parallel plate condensers of vanishing thickness placed along the positive and negative \( x \) and \( y \) axes at equal distances from the origin and joined in series, this being a consequence of the additivity of the inverses of the capacitances. This, however, is outside the scope of our analysis. It is of interest, too, to point out that the improved lower bound \( C/C_0 \geq \sqrt[\mu_1 \mu_2} \) holds for capacitors consisting of two conductors (at different potentials) in the form of similarly oriented concentric equilateral triangles, similarly oriented concentric squares, and of concentric circles, the orientation of the coordinate axes being immaterial in all cases. This can be proved by simple coordinate transformations together with the techniques of symmetrization [4]. It is reasonable to conjecture that the above mentioned bound holds.