A study has been made of the movement of ions in a cyclotron with azimuthal variation of the magnetic field of the sector type. Methods are given which can be used to calculate the period of rotation, the frequencies of the vertical and horizontal oscillations and also the least required voltage on the dees. The method of calculating the voltage is also suitable for an ordinary cyclotron.

Introduction

It was shown in [1] that if the magnetic field of a cyclotron has the form \( H_z = H_0 \times (1 + Ar \cos n \varphi + Br^2) \), then the vertical stability is not disturbed, despite the increase in the mean value of the field intensity with increase in the radius. E. M. Moroz and M. S. Rabinovich [2] showed that the acceleration of ions in a cyclotron with a radially increasing magnetic field will be stable if the field changes abruptly along the azimuth. The creation of any field changing according to a previously given law causes considerable technical difficulties. However, this is not required since the necessary conditions for acceleration, i.e., constancy in the period of rotation and stability of movement, can be achieved under the most widely differing concrete forms of the field. In 1957 the authors of this article analyzed the movement of ions in a magnetic field changing according to an arbitrary law both along the radius and along the azimuth. It was shown that in the important practical cases the effect of the highest harmonics of the field is small; therefore in this article the main attention was paid to the case of cosine change in the field.

1. Closed Orbit and Period of Rotation of Ions. We will assume that in a cylindrical system of coordinates the magnetic field intensity \( H_z \) in the middle plane between the poles \((z = 0)\) can be represented in the form

\[
H_z(r, \varphi) = H_0 \left[ 1 + a F(r) \cos n \varphi + \beta f(r) \right], \tag{1}
\]

where \( H_0 \) is the magnetic field intensity in the center of the cyclotron chamber; \( n \) is a whole number characterizing the periodicity of change in field with \( 0 < n \) and \( \beta < 1 \) are parameters which are independent of \( r \) and \( \varphi \), introduced for convenience in the subsequent calculations; \( F(r) \) and \( f(r) \) are certain functions of \( r \).

The curvature of the orbit \( K \) at each point is connected with the magnetic field intensity \( H_z \) by the known relationship

\[
K = \frac{1}{v} \frac{e H_z}{m c} = \frac{1}{r_1} \frac{H_z}{H_0}, \tag{2}
\]

where \( r_1 = mcv/eH_0 \) is the radius of the circle along which a particle would move at a speed \( v \) in a homogeneous field \( H_0 \).

\( m = \sqrt{1 - \frac{v^2}{c^2}} \) is the mass of the moving particle \((m_0 \) is the rest mass of the particle); \( c \) is the speed of light; \( e \) is the charge on the particle.

From expressions (2) and (1) it follows that

\[
K r_1 = \frac{\frac{e^2}{r_1^2} + 2 \frac{e^2 r^2}{r_1^4} - \frac{e^2}{r_1^4}}{\left( \frac{e^2}{r_1^2} + \frac{e^2}{r_1^2} \right)} = 1 + a F(r) \cos n \varphi + \beta f(r). \tag{3}
\]

Here we used the known expression for curvature; the stroke indicates differentiation with respect to \( \varphi \).
In this section we will consider the solution $r(\varphi)$ of the differential equation (3) which describes an orbit which is closed in one rotation and will be later designated by $R(r)$. The free oscillations relative to the closed orbit will be considered in the next section.

As $\alpha$ and $\beta$ tend to zero, $H_2$ tends to $H_0$ and the closed orbit is pulled toward a circle with radius $r_1$ and a center at the origin of the coordinate system. The expression for the radial coordinate $R(\varphi)$ of a closed orbit can therefore be sought in the form of the following series in terms of $\alpha$ and $\beta$:

$$ R = r_1 \left[ 1 + \sum_{i, k=0}^{\infty} a^i \beta^k \chi_{ik} (r_1, \varphi) \right], $$

(4)

where $\chi_{ik}$ are unknown functions of $r_1$ and $\varphi$. The value $\chi_{00}$ written in this way should presumably be equal to zero.

We will substitute in equation (3) $r = R$ from (4) and expand the left and right sides of the equality into series in terms of $\alpha$ and $\beta$. The obtained equality should be satisfied identically for any values of $\alpha$ and $\beta$. Therefore, equating the coefficients with the same powers of $\alpha$ and $\beta$ in both parts of the equality, it is possible to obtain a series of differential equations to determine $\chi_{ik}$:

$$ \begin{align*}
\chi_{10}'' + \chi_{10} &= -F(r_1) \cos n\varphi; \\
\chi_{01}'' + \chi_{01} &= -f(r_1); \\
\chi_{20}'' + \chi_{20} &= \frac{1}{2} \chi_{10}'' + 2 \chi_{10}' \chi_{10} - r_1 \frac{dF}{dr} \chi_{10} \cos n\varphi; \\
\chi_{11}'' + \chi_{11} &= 2 \chi_{10} \chi_{01} + 2 \chi_{10}' \chi_{01}' + 2 \chi_{10}' \chi_{01}''; \\
\chi_{21}'' + \chi_{21} &= -r_1 \frac{dF}{dr} \chi_{10} \cos n\varphi - r_1 \frac{df}{dr} \chi_{01} \cos n\varphi; \\
\chi_{02}'' + \chi_{02} &= \chi_{01}'' + \frac{1}{2} \chi_{01}'' + 2 \chi_{01}' \chi_{01}''; \\
\chi_{12}'' + \chi_{12} &= -r_1 \frac{dF}{dr} \chi_{01} \cos n\varphi - r_1 \frac{df}{dr} \chi_{01} \cos n\varphi; \\
\chi_{22}'' + \chi_{22} &= -r_1 \frac{dF}{dr} \chi_{01} \cos n\varphi - r_1 \frac{df}{dr} \chi_{01} \cos n\varphi; \\
\chi_{03}'' + \chi_{03} &= \chi_{02}'' + \frac{1}{2} \chi_{02}'' + 2 \chi_{02}' \chi_{02}''; \\
\chi_{13}'' + \chi_{13} &= -r_1 \frac{dF}{dr} \chi_{02} \cos n\varphi - r_1 \frac{df}{dr} \chi_{02} \cos n\varphi;
\end{align*} $$

(5)

Since the magnetic field's dependence on $\varphi$ has a period $2\pi/n$, then when solving the linear differential equations (5) allowance should be made only for the partial solution, rejecting the general solutions of equations without a right side (with period $2\pi$) as not having a physical meaning:

$$ \begin{align*}
\chi_{10} &= \frac{F(r_1)}{n^2 - 1} \cos n\varphi; \\
\chi_{11} &= -F(r_1); \\
\chi_{20} &= F^2(r_1) \left[ \frac{2 - 3n^2 - 2 (n^2 - 1) Q}{4 (n^2 - 1)^2} + \frac{5n^2 - 2 + 2 (n^2 - 1) Q}{4 (n^2 - 1)^2} \right] \cos 2n\varphi; \\
\chi_{12} &= -F(r_1) f(r_1) \left( 2 + Q - \frac{4}{n^2 - 1} \frac{r}{f} \frac{df}{dr} \right) \cos n\varphi; \\
\chi_{03} &= f^2(r_1) \left( 1 + \frac{r}{f} \frac{df}{dr} \right) \cos n\varphi;
\end{align*} $$

(6)

where $Q = (r/F)(df/dr)$. Determining the value of $\chi_{ik}$ in this way using expression (4) it is possible to calculate $R$ with any degree of accuracy.

The order of smallness of the terms in expression (4) can be evaluated if we use the condition of stability of movement along the vertical (see section 2) from which it follows that

$$ \alpha^2 F^2(r) \sim \beta f(r). $$

(7)

On the other hand, to compensate the relativistic effect it is essential that

$$ \beta f(r) \sim \frac{v^2}{c^2}. $$

(8)

Considering expressions (7) and (8), we find with an accuracy up to terms of the order $v^2/c^2$ that

$$ R = r_0 \left( 1 + \frac{\alpha F^2}{2c^2} + \alpha \chi_{10} + \beta \chi_{01} + \alpha^2 \chi_{20} \right), $$

(9)