A method for finding the reduction of the direct products of two irreducible representations of SU(4) is formulated by tensor method. This formula is applied for some particular cases.

1. INTRODUCTION

In very recent months, after the discovery of $\psi$ resonances [1], SU(4) symmetry charm scheme for hadron classification has become very topical [2]. The purpose of the present work is to develop a method for calculation of the reduction of direct products of two irreducible representations of SU(4). As its rank is 3, three members of the Lie algebra can be diagonalised simultaneously. Thus the irreducible representations are labelled accordingly by 3 non-negative integers $\lambda_1, \lambda_2$ and $\lambda_3$. The dimension of the representation is given by

$$D = \frac{1}{12} (\lambda_1 + 1) (\lambda_2 + 1) (\lambda_3 + 1) (\lambda_1 + \lambda_2 + 2) (\lambda_2 + \lambda_3 + 2) (\lambda_1 + \lambda_2 + \lambda_3 + 3).$$

Let there be another unitary irreducible representation $\lambda'_1, \lambda'_2, \lambda'_3$. We have to find the unitary irreducible representation occurring in the Clebsh-Gordan series, which is symbolically represented on the right-hand side of the expression

$$\sum_{\alpha\beta\gamma} (\lambda_1, \lambda_2, \lambda_3) \otimes (\lambda'_1, \lambda'_2, \lambda'_3) = \sum_{\alpha\beta\gamma} (\xi_{\alpha\beta\gamma}).$$

We shall use the characterization of the irreducible representation of SU(4) as the transformation induced on irreducible tensorial sets by unitary unimodular transformation, in the complex vector space bounded by hyper-planes of SU(4).

2. IRREDUCIBLE TENSOR

Now we introduce the concept of an irreducible tensor. A tensor which transforms according to an irreducible representation of SU(4) is an irreducible tensor. That is, an irreducible tensor is one whose sub-space contains no invariant sub-spaces. It is obvious that, to understand the properties of group representations, we need only consider irreducible representations; since by definition any irreducible tensor splits into a sum of irreducible representations. Hence its space is made up of several invariant sub-spaces. In this way we can decompose the direct products of irreducible representations into direct sums of irreducible representations.
To achieve this we take our irreducible tensor $x^{\lambda_1, \lambda_2}_{\lambda_3}$ with $\lambda_1, \lambda_2$ upper indices (contravariant) and $\lambda_3$ lower index (covariant). We shall denote $x^{\lambda_1, \lambda_2}_{\lambda_3}$ by $(\lambda_1, \lambda_2, \lambda_3)$ which stands for unitary irreducible representation of $SU(4)$. Similarly let $y^{\lambda_1', \lambda_2'}_{\lambda_3'}$ be another irreducible tensor which is denoted symbolically by $(\lambda_1', \lambda_2', \lambda_3')$. This also represents the unitary irreducible representation of $SU(4)$.

We use 3 invariant symmetric and anti-symmetric tensors $\delta^a_b$, $\epsilon_{abcd}$ and $\epsilon^{abcd}$ for constructing tensors belonging to unitary irreducible representations that appear on the right-hand side of (2).

The following processes are used to expand tensors into their irreducible parts.

1. Contract an upper index of $x$ with lower index $y$ any number of times, using $\delta$'s.
2. Contract a lower index of $x$ with an upper index of $y$ any number of times using $\delta$'s.
3. Contract an upper index of $x$ and an upper index of $y$ any possible number of times, using $\epsilon_{abcd}$ whereby each time, the number of lower index gets increased by one and an upper index decreases by two.
4. Contract a lower of $x$ and a lower index of $y$ using $\epsilon^{abcd}$ any possible number of times, whereby each time, a lower index gets decreased by two and an upper index gets increased by one. After these constructions, the process of symmetrization and removal of non-zero traces are to be performed. Further, processes 3 and 4 cannot be used simultaneously, because we cannot use $\epsilon_{abcd}$ and $\epsilon^{abcd}$ at the same time.

To get unitary irreducible representation on the right-hand side of (2), let us suppose that process 1 has been used $p$ times on $\lambda_1$ and $p'$ times on $\lambda_2$ whereas process 2 is used $q$ times on $\lambda_3$ using $\lambda_1$ and $q'$ times using $\lambda_2$.

Then the possible values of $p, p', q$ and $q'$ are

$$
\begin{align*}
0 & \leq p \leq \lambda_1 \\
0 & \leq p' \leq \lambda_2 \\
0 & \leq q \leq \lambda_3 \\
0 & \leq q' \leq \lambda_3
\end{align*}
$$

For a definite allowed choice of $p, p', q$ and $q'$, two sets of unitary irreducible representations are to be considered, depending on whether we use process 3 or 4.

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We agree that the possibility that neither process 3 nor 4 is used, be included in this set.

Let the process 3 be used $n$ times, then $n$ lies between

$$
0 \leq n \leq \min (\lambda_1 + \lambda_2 - p - p'; \lambda_1' + \lambda_2' - q - q'),
$$

where $\min (A, B)$ stands for the smaller of integers $A$ and $B$.