Finite Elastic Deformations of Bodies with Irregular Shapes

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1. Introduction

In the theory of finite deformation of elastic solids, problems for which exact solutions are available are limited to bodies whose unstrained and strained states are bounded by co-ordinate surfaces of the well known co-ordinate systems and therefore possess a considerable degree of symmetry (e.g. see Green and Zerna [1]). The object of this paper is to develop a method of obtaining finite deformation solutions for bodies whose undeformed shape is a perturbation of the undeformed shape of another body for which a finite deformation solution is already available. The theory is based in part upon the theory of small deformations superimposed on finite elastic deformations that was given by Green, Rivlin and Shield [2] and which is recounted in Green and Zerna [1]. Other work in this area was done by Kydoniefs and Spencer [3] and Kydoniefs [4] who treated problems of inflation under uniform internal pressure.

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and rotation about its axis of symmetry of a torus of incompressible elastic material by reducing them to a perturbation of the problem of extension and inflation of a circular tube. Apart from this Spencer [5, 6] has developed and illustrated a theory of finite deformations with a perturbed strain energy function while the possibility of the present investigation was alluded to in a survey article by Adkins [7].

A summary of the notation and certain pertinent equations of finite deformation theory for elastic bodies is given in section 2. In section 3 a consideration of the effect of replacing the surface \( f = 0 \) that bounds the unstrained body by a surface \( f + \varepsilon g = 0 \), where \( f \) and \( g \) are given functions, is presented. A finite deformation, that is consistent with definite tractions on the surface \( f = 0 \), is assumed to be known throughout the whole of space. However, this deformation will not in general produce the required prescribed tractions on the surface \( f + \varepsilon g = 0 \). In order to satisfy the traction boundary condition on this surface an additional deformation is superposed on the given deformation. It is assumed that the additional displacements that occur are of order \( \varepsilon \) and then the development of the theory closely follows the theory of small deformations superimposed on large deformations given in Green and Zerna [1]; the main difference being that the resulting tractions are evaluated and the boundary conditions satisfied on the surface \( f + \varepsilon g = 0 \).

In section 4 the finite deformation solution for the extension and torsion of a circular cylinder is presented while in section 5 this finite deformation is used as a first approximation in determining a solution to the problem of finite extension and torsion of an elliptic cylinder the quotient of whose major axis by minor axis is \( (1 + \varepsilon) \). It is found that the deformation may be maintained by a resultant force and couple acting parallel to the axis of the cylinder and for small twist the results are in agreement with those predicted by the theory of small twist superimposed on finite extension of an arbitrary cylinder that is given in Green and Zerna [1].

In section 6 the technique developed in this paper is used to facilitate an analysis of the problem of finitely extending an axisymmetric body that is (i) a circular cylinder at all points outside a strip of finite length where its undeformed shape is a perturbation of the circular cylinder, (ii) obtained by rotating about the axis of symmetry a curve described by the sine function.

2. Notation and Formulae for Finite Deformation

The notation we shall employ is that of Green and Zerna. [1], where a full account of the derivation of finite deformation theory may be found. An isotropic and homogeneous elastic body with a strain energy function \( W \) is denoted by \( B_0 \) and \( B \) in its unstrained and finitely deformed states respectively. The points of \( B_0 \) are referred to a rectangular cartesian co-ordinate system \( x_i \), and the points of \( B \) to a rectangular cartesian co-ordinate system \( y_i \). Curvilinear co-ordinates \( \theta_i \), which are convected with the body, are then related to \( x_i \) and \( y_i \) by

\[
x_i = x_i(\theta_1, \theta_2, \theta_3), \quad y_i = y_i(\theta_1, \theta_2, \theta_3).
\]

The covariant and contravariant metric tensors of the \( \theta_i \) co-ordinates are denoted by \( g_{ij}, g^{ij} \) respectively in \( B_0 \), \( G_{ij}, G^{ij} \) respectively in \( B \), while the covariant and contravariant base vectors of \( B \) are designated by \( G_i \) and \( G^i \) respectively.