This paper describes the class of infinite horizon linear programs that have finite optimal values. A sequence of finite horizon (T period) problems is shown to approximate the infinite horizon problems in the following sense: the optimal values of the T period problems converge monotonically to the optimal value of the infinite problem and the limit of any convergent subsequence of initial T period optimal decisions is an optimal decision for the infinite horizon problem.

1. Introduction

This paper examines long range planning models that can be presented as linear programs over an infinite planning horizon. The main results characterize problems that have finite optimal values and establish procedures for approximating optimal solutions by solving a T period linear program. Three possible approaches to this task are examined; in general, only one procedure leads to solvable T period problems. The T period problem is designed by decoupling the infinite problem into the sum of a T period problem and another infinite problem that commences at time T; call these problems 1 and 2. Any feasible solution of problem 1 produces an input into problem 2. The scheme calculates an approximate salvage value of the input from problem 1 to problem 2. This salvage value is then included in the objective of problem 1. As T increases the error in calculating the salvage value becomes less and less significant. The main conclusions of the paper are easily stated:

(1) The infinite horizon problems are solved in the following sense. Let $V^T$ be the optimal value of the T period approximation, and let $x_0^T$ be the time 0 optimal decision for the T period problem. $V^T$ decreases to the optimal value of the infinite problem and any limit of the sequence $x_0^T$ is an optimal initial decision for the infinite problem.

(2) Duality is not the primary consideration in the study of infinite horizon linear programs.

(3) The calculation of an equilibrium optimal policy, if one in fact exists, does not, in general, assist in the solution of a discounted criterion problem.

The remainder of this section introduces the problem, summarizes results, relates this work to others, and describes the notational conventions used in the paper.
Time is discrete \( t = 0, 1, 2, \ldots \). At any time \( t \) the state of the system is \( s_t \). Decisions \( x_t \) are constrained by the relations \( Ax_t = s_t, \ x_t \geq 0 \). If decision \( x_t \) is taken, then a reward with time zero value \( \alpha^T p x_t \) is received and the state becomes \( Ax_{t+1} = s_{t+1} = b + K x_t \). \( A \) and \( K \) are \( m \times n \) matrices, \( p \) and \( n \)-vector, \( b \) an \( m \)-vector, \( s_t \) an \( m \)-vector and \( \alpha \) a positive scalar less than 1.

If the initial (time zero) state is \( s_0 = s \) then the infinite sequence of decisions \( \{x_t\} \) is constrained by

\[
\begin{align*}
Ax_0 &= s, \quad x_0 \geq 0, \\
Ax_t &= b + K x_{t-1}, \quad x_t \geq 0, \quad t \geq 1.
\end{align*}
\] (1.1)

Let \( X(s) \) be the set of \( \{x_t\} \) that satisfy (1.1). For any \( \{x_t\} \) define

\[
p(\{x_t\}) = \lim \inf_{T \to \infty} \sum_0^T \alpha^t p x_t.
\] (1.2)

The optimization problem under consideration is

\[
\begin{align*}
\text{maximize} \quad & p(\{x_t\}), \\
\text{subject to} \quad & \{x_t\} \in X(s).
\end{align*}
\] (1.3)

Let \( e \) be an \( n \)-vector of ones, a summation vector. A sequence \( \{x_t\} \) is called \( \alpha \)-convergent if the increasing sequence \( \sum_0^T \alpha^t e x_t \) converges to a finite limit.

Denote \( X_\alpha(s) \) as the set of \( \alpha \)-convergent \( \{x_t\} \in X(s) \), and note that \( p(\{x_t\}) = \sum_0^\infty \alpha^t p x_t \) for \( \{x_t\} \in X_\alpha(s) \).

The formulation is slightly more general than it appears. Inequalities can be included using slack or surplus variables. In addition, if the data are \( \bar{\alpha}, \bar{K} \) and \( b_t = \theta^t b \), with decision variables \( y_t \), then one can form an equivalent problem with \( x_t = y_t / \theta^t \), \( \alpha = \bar{\alpha} \theta \), and \( K = \bar{K} / \theta \).

Section 2 presents assumptions made in the paper and comments on their immediate consequences and the problem of verification. These assumptions hold throughout and are not restated for each result. Section 3 defines the optimal value of (1.3) as a function of the initial state \( s_0 = s \). The consequences of the assumptions not holding are investigated in Sections 3 and 4. Section 4 contains an important result: if a program \( \{x_t\} \) is feasible but not \( \alpha \)-convergent, then \( \{x_t\} \) is a bad program in the sense that \( p(\{x_t\}) = -\infty \).

Section 5 describes a sequence of solvable \( T \) period linear programs that can be used to solve (1.3). The optimal values of the \( T \) period problems decrease to the optimal value. A subsequence of the optimal initial decisions converge to an optimal initial decision of the infinite problem. Section 6 presents sufficient optimality criteria and comments on duality and the establishment of necessary conditions for optimality. Section 7 examines the question of calculating optimal equilibrium policies; i.e., \( x_t = x \) for all \( t \). The appendix contains the statement and proof of two lemmas that are used in the main body of the study.

This paper is based on previous work by Manne [13], Hopkins [11, 12], Hopkins and Grinold [8], and Evers [3]. The work of Evers [3] motivated this study and is the genesis for the important assumption II in Section 2. Assumptions like II are implicitly made in [13] and [8], however, Evers was the first to