An Upper Bound for a Percolation Constant

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Ordinary percolation theory as first discussed in [1] is a model for the flow of a fluid through a randomly dammed medium, and is closely connected with the distribution of cluster sizes in random mixtures (see [2]). In [3] this model is generalised to consider the case in which each bond of the medium instead of being closed or open has assigned to it a non-negative random variable, independently and identically distributed for each bond. This random variable might in a typical application represent the time taken for the fluid to pass along the bond and we are interested in the rate at which fluid spreads from the source to the atoms of the medium.

In the note we consider only the case where the underlying medium is the square lattice of integer coordinated atoms \((x, y)\) and bonds (all of unit length) parallel to the \(x\)- and \(y\)-axes and unoriented. To each bond of this lattice we independently assign a time coordinate which is a random variable drawn from the uniform rectangular distribution on \((0, 1)\). The time coordinate of any lattice path is defined to be the sum of the random variables assigned to its component bonds, and it may therefore be regarded as the time taken by a fluid to flow along the path. We may now define the first passage time between any two atoms of the lattice to be the minimum time it would take for the fluid to travel between the two atoms.

The first passage time \(t_n(\omega)\) between the origin and \((n, 0)\) is obviously a random variable depending on \(n\) and the configuration of time coordinates \(\omega\). Clearly \(\{t_n(\omega)\}_{n=1}^\infty\) is a stochastic process on the well defined probability space of all possible configuration of time coordinates.

If we let \(\tau(n) = E t_n(\omega)\), it is not difficult to show that

\[
\tau(n) + \tau(m) \geq \tau(m + n)
\]

and also that

\[
0 \leq \tau(n) \leq \frac{n}{2}
\]

and thus by the theory of subadditive functions there exists a constant \(\mu\) such that

\[
\inf_n n^{-1} \tau(n) = \lim_{n \to \infty} n^{-1} \tau(n) = \mu \leq \frac{1}{2}.
\]

This constant \(\mu\) is of fundamental importance in first passage percolation theory but so far it has been impossible to evaluate it theoretically. Indeed the best result so far [3] is that

\[
0 \leq \mu \leq 0.425.
\]

In this note we reduce this upper bound by a Monte Carlo experiment on the Ferranti Mercury Computer at the Oxford University Computing Laboratory.

The procedure involved using a pseudo random number sequence to simulate the flow of fluid through the lattice, and thus obtain a realisation of the process \(\{t_n(\omega)\}_{n=1}^\infty\). Despite the size of the problem, we obtained 100 independent realisations of the process \(\{t_n(\omega)\}_{n=1}^{40}\). Figure 1 shows the estimates of \(n^{-1} \tau(n)\) obtained and combining them with (3) we obtained that with 95% confidence

\[
0 \leq \mu \leq 0.35031.
\]

This upper bound can, however, be further reduced by using the fact (proved in [3]) that if \(s_n(\omega)\) denotes the first passage time from the origin to the set of atoms on \(x = n\), then although

\[
\psi(n) = E s_n(\omega) \leq \tau(n)
\]

1) Numbers in brackets refer to References, page 522.
we do have that
\[ \lim_{n \to \infty} n^{-1} \psi(n) = \inf_{n \to \infty} n^{-1} \psi(n) = \mu \tag{8} \]

Figure 2 shows estimates of \( n^{-1} \psi(n) \) obtained from the same experiment. Combining these with (8) we can therefore say that with 95\% confidence
\[ \mu < 0.32812 \tag{9} \]

At this stage it is impossible to decide how close this upper bound is to the true value. The main difficulty is that we know no way of obtaining a non-trivial lower bound for \( \mu \).

Finally we mention the results displayed in Figure 3. They suggest that \( \text{var} \, t_n(\omega) \) is \( o(n) \) as \( n \to \infty \). A proof of this would be of great interest theoretically, particularly with regard to the convergence properties of the process \( \{n^{-1} t_n(\omega)\} \) as \( n \to \infty \). So far, however, we have only been able to prove that \( \text{var} \, t_n(\omega) \) is \( o(n^2) \).