On the Eigenvalues of the Matrix Pencil $A + \mu B$

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1. Introduction

Only in very special cases can the eigenvalues of the matrix pencil $A + \mu B$ be related to those of $A$ and $B$ separately. For example if $A$ and $B$ commute they can be reduced to triangular form by one and the same similarity transformation and the eigenvalues of $A + \mu B$ are $\alpha_i + \mu \beta_i$, where $\alpha_i$ and $\beta_i$ are the eigenvalues of $A$ and $B$, respectively, ordered in the appropriate way [8]. But in general no simple relationship exists and repeated computation is required to determine the eigenvalues when the parameter $\mu$ is varied. Herein are reported results that simplify parametric studies of the eigenvalues as $\mu$ varies when $A$ and $B$ are fixed; some applications are described in [1], [5] and [6].

The objective can be explained with concepts from the classical theory of algebraic invariants [2]. If a scalar-valued function $f$ of one or more $n \times n$ matrices $A, B, C, \ldots$ remains unchanged when $A, B, C, \ldots$ are transformed by any arbitrary similarity transformation, then $f$ is said to be a scalar invariant of $A, B, C, \ldots$ with respect to the group $GL(n)$ of invertible linear transformations. Moreover, if $f$ is a polynomial in the elements of $A, B, C, \ldots$, then there exists a finite set of invariants of $A, B, C, \ldots$, the so-called integrity basis, such that $f$ can be expressed as a polynomial in these invariants [2], [10]. Clearly the determinant $d(A, B, I) = |\lambda I - A - \mu B|$ is a polynomial scalar invariant of $A, B$ and $I$ and can consequently be expressed as a polynomial in the invariants of $A, B$ and $I$. Our purpose here is to determine what number of invariants from an integrity basis is required for the representation of $d(A, B, I)$, and hence of the eigenvalues $\lambda$, and to display these invariants for a number of cases.

Integrity bases for scalar-, vector- and tensor-valued functions have received much attention in the literature on continuum mechanics but the results are always obtained for three-dimensional space and usually only for the orthogonal group [7], [8]. Moreover, the invariants have been obtained by enumerating possible cases and no independent means of predicting the total number of invariants has been available.

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2. Determination of the Number of Invariants

Let \( A \) and \( B \) be arbitrary \( n \times n \) matrices over the complex field and let \( \mu \) be an arbitrary complex number. The eigenvalues of \( C = A + \mu B \) satisfy the characteristic equation

\[
|\lambda I - C| = 0 = \lambda^n + c_1(\mu) \lambda^{n-1} + \cdots + c_n(\mu)
\]

where \( c_k(\mu) \) is equal to \((-1)^k\) times the sum of the principal minors of order \( k \) of \( C \). Because \(|\lambda I - C|\) is an invariant of \( \lambda I - C \) with respect to \( GL(n) \) the coefficients \( c_k(\mu) \) and the eigenvalues \( \lambda \) are also invariants of \( \lambda I - C \) with respect to \( GL(n) \). In the following section we derive expressions for the \( c_k(\mu) \) in terms of the invariants of \( A \) and \( B \) and of their products. Here we determine the number of independent invariants on which the \( c_k(\mu) \) depend.

The \( c_k(\mu) \) can alternatively be expressed in terms of the traces of the first \( n \) powers of \( C \). The definitions

\[
t_k = (-1)^k c_k(\mu) \quad (1)
\]

and

\[
T_k = \text{trace } C^k = \text{trace } (A + \mu B)^k \quad (2)
\]

lead to the formulas [4]

\[
c_0 = 1 \quad \text{and for } k \geq 1 \quad c_k(\mu) = (-1)^k t_k = -\frac{1}{k} \sum_{i=1}^{k} c_{k-i} T_i.
\]

From (2) and (3) it follows that the coefficients \( c_k(\mu) \) are polynomials of degree \( k \) in \( \mu \), the coefficients of which are the traces of products of \( A \) and \( B \) of degree \( k \) or less. Therefore the number of invariants of \( A, B \) and their products which appear in each \( c_k(\mu) \) is given by the total number of invariants in the sequence

\[
T_1, \ldots, T_{k-1}, T_k.
\]

Furthermore, because all invariants that appear in the first \( k - 1 \) coefficients also appear in \( c_k(\mu) \), only the latter need be considered.

Consider the term

\[
T_k = \text{trace } (A + \mu B)^k = \sum_{s=0}^{k} \mu^s \text{trace } \left\{ \begin{array}{cc} A & B \\ k-s & s \end{array} \right\}
\]

where the bracket \( \left\{ \begin{array}{cc} x & y \\ p & q \end{array} \right\} \) in the noncommutative variables \( x \) and \( y \) denotes the sum of all possible products of \( px \)'s and \( qy \)'s. (For example, \( \left\{ \begin{array}{cc} x & y \\ 2 & 1 \end{array} \right\} = x^2 y + xyx + yx^2 \) [11].)

Even if \( AB \neq BA \), still \( T_1^{AB} = T_1^{BA} \), and in general all products of \( A \) and \( B \) of the form

\[
A^{\sigma_1} B^{\sigma_2} A^{\sigma_3} B^{\sigma_4} \cdots A^{\sigma_k} B^{\gamma_k}, \sigma_1 + \cdots + \sigma_k + \gamma_1 + \cdots + \gamma_k = k
\]