Stress Functions for Plane Problems with Couple Stresses

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1. Introduction

For plane problems and in the absence of body forces and body couples, the stress equations of equilibrium for a continuum which can support couple stresses may be written as

\[
\frac{\partial t_{xx}}{\partial x} + \frac{\partial t_{yx}}{\partial y} = 0, \quad \frac{\partial t_{xy}}{\partial x} + \frac{\partial t_{yy}}{\partial y} = 0, \quad \frac{\partial m_{xz}}{\partial x} + \frac{\partial m_{yz}}{\partial y} + t_{xy} - t_{yx} = 0
\]

when referred to rectangular Cartesian coordinates. In (1.1) \(t_{xx}, t_{yx}, t_{xy}, \) and \(t_{yx}\) are the components of the stress tensor, and \(m_{xz}\) and \(m_{yz}\) are the components of the couple stress tensor. The domain of these functions is some bounded region \(R\) of the \(x, y\)-plane. A direct and elementary derivation of (1.1) has been given by Mindlin [1]. They can also be reached by specialization of the corresponding equations for the three-dimensional case [2].

It is the purpose of the present paper to give the general solution of (1.1) in terms of arbitrary functions (stress functions). In the non-polar case (identically zero couple stresses), the well-known general solution was given by Airy [3]. It is important to emphasize that since we are dealing only with the stress equations of equilibrium, all of our results (except those in Section 4) are independent of any constitutive equations which the stresses may be required to satisfy.

In Section 2 the stress function solution is derived and shown to be complete. The degree of arbitrariness of the stress functions for a given set of stresses is then examined.

In Section 3 the stress functions are interpreted in terms of the resultant force and moment transmitted across an arc in the body. This leads to necessary and sufficient conditions for the stresses to satisfy in order that the stress functions be single-valued.

Finally, in Section 4 the stress function solution given by Mindlin [1] for the case of linearized elasticity is obtained from our general solution.

2. Stress Function Solution

Here and in what follows, we will not state explicitly the smoothness requirements. They may be readily inferred from well-known theorems of calculus; see, for example, Courant [4].
The following theorem, which supplies the stress function solution of (1.1), may be confirmed by direct substitution.

Theorem 2.1. Define stresses through
\[
\begin{align*}
    t_{xx} &= \frac{\partial F}{\partial y}, & t_{yy} &= -\frac{\partial G}{\partial x}, & t_{xy} &= -\frac{\partial G}{\partial y}, & t_{yx} &= \frac{\partial F}{\partial x}, & m_{xz} &= \frac{\partial H}{\partial y} - F, & m_{yz} &= -\frac{\partial H}{\partial x} - G,
\end{align*}
\]
where \( F, G, \) and \( H \) are arbitrary functions. Then these stresses satisfy (1.1).

The proof of the next theorem, which shows that every solution of (1.1) can be represented in the form (2.1), is actually the motivation for representation (2.1).

Theorem 2.2. Let the stresses satisfy (1.1). Then there exist functions \( F, G, \) and \( H \) on \( R \) such that the stresses can be represented by (2.1). Furthermore, if \( R \) is simply connected, then the stress functions \( F, G, \) and \( H \) will be single-valued.

Proof. According to the theory of total differentials [4], (1.1) implies the existence of a function \( F \) (single-valued if \( R \) is simply connected) such that
\[
t_{xx} = \frac{\partial F}{\partial y}, \quad t_{xy} = -\frac{\partial F}{\partial x}.
\]
Similarly, by (1.1) there exists a function \( G \) such that
\[
t_{xy} = \frac{\partial G}{\partial y}, \quad t_{yy} = -\frac{\partial G}{\partial x}.
\]
Then (1.1) can be written as
\[
\frac{\partial}{\partial x} (m_{xz} + F) + \frac{\partial}{\partial y} (m_{yz} + G) = 0,
\]
and hence there is a function \( H \) such that
\[
m_{xz} + F = \frac{\partial H}{\partial y}, \quad m_{yz} + G = -\frac{\partial H}{\partial x}.
\]
This completes the proof.

It is of some interest to know to what extent the stress functions are determined by the stresses they represent. This information is contained in the following theorem.

Theorem 2.3. Let a given set of stresses which meet (1.1) be represented according to (2.1) by the stress functions \( F, G, H \) and also by the stress functions \( F', G', H' \). Then
\[
F - F' = F_0, \quad G - G' = G_0, \quad H - H' = -G_0 x + F_0 y + H_0,
\]
where \( F_0, G_0, H_0, \) are constants.

Proof. By (2.1)
\[
\frac{\partial}{\partial x} (F - F') = \frac{\partial}{\partial y} (F - F') = 0, \quad \frac{\partial}{\partial x} (G - G') = \frac{\partial}{\partial y} (G - G') = 0,
\]
\[
\frac{\partial}{\partial x} (H - H') = -(G - G'), \quad \frac{\partial}{\partial y} (H - H') = (F - F').
\]
Hence
\[
F - F' = \text{const.} = F_0, \quad G - G' = \text{const.} = G_0,
\]
\[
H - H' = -G_0 x + F_0 y + \text{const.} = -G_0 x + F_0 y + H_0;
\]
and the proof is complete.

It is worth noting that (2.1) may be regarded as a special case of Günther's [5, 6] stress function solution of the three-dimensional equilibrium equations. Also, it is a trivial matter to write (2.1) in invariant form.