A Note on the Mechanical Constitutive Equations for Materials with Memory

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1. Introduction

In obtaining the constitutive equations for materials with memory we consider a time-dependent deformation of the body in which a particle initially at $X_0$ moves to $x_i(t)$ in the system at time $t$. The stress components $\sigma_{ij}$ at time $t$, in the system $x$, are assumed to be func-
tionals of the deformation gradients $x_{p,q}(t)$ in the interval $[-\infty, t]$, thus:

$$\sigma_{ij} = F_{ij} \left[ x_{p,q}(t) \right].$$

From the fact that simultaneous rotation of the body and the reference system leaves the stress unaltered, it has been shown (Green and Rivlin, 1957) that the functionals $F_{ij}$ must satisfy the following relation:

$$F_{ij} \left[ \bar{x}_{p,q}(t) \right] = a_{ik} a_{lj} F_{kl} \left[ x_{p,q}(t) \right].$$

where

$$\bar{x}_p(t) = a_{pq}(t) x_q(t), \quad a_{pq} = a_{q,p}(t)$$

and $a_{ij}(t)$ is an arbitrary time-dependent proper orthogonal tensor.

It has been shown in a variety of ways that the relation (1.2) implies that the functionals $F_{ij}$ must be expressible in the form

$$F_{ij} = X_i, A, X_j, B G_{p,q}(t),$$

where

$$G_{p,q}(t) = \sum \sum x_k, p x_k, q \cdot$$

The method employed by Green and Rivlin (1957) to obtain this result implicitly assumes that the functionals $F_{ij}$ are continuous. This is not a severe restriction from a physical standpoint since all physical phenomena are in fact continuous and it is only the restriction on the form of the functionals that is relevant. The mathematical idealization to discontinuous functionals may be introduced after the functional representation theorem has been employed rather than before. However, the later and more elegant derivations of (1.4) due to Noll (1958) and to Pipkin and Rivlin (1961) do not make the mathematically restrictive assumption that the functionals are continuous. They both depend on making particular choices of the rotation $a_{ij}(t)$. It is the object of the present note to unify, and perhaps deepen, these two derivations. This is done with the aid of a theorem which is proven in the next section.

2. The Basic Theorem

Consider a deformation in which a generic particle of a body, which is at $X_A$ in a rectangular Cartesian coordinate system $x$ at a reference time $t_0$, moves to $x_i$ in the same system at time $t$. Let $A = \| a_{ij} \|$ be a rotation tensor each of whose components is a function of the deformation gradients $x_{p,q}$. Thus,

$$a_{ij} = \bar{f}_{ij}(x_{p,q}).$$

We shall show that $a_{ij}$ is expressible in the form

$$a_{ij} = h_{ij}(G_{p,q}) x_{i,A},$$

where

$$G_{p,q} = \sum \sum x_{k,p} x_{k,q} \cdot$$

Let $\bar{x}$ be a second coordinate system which coincides with the system $x$ at time $t_0$. Let $\bar{x}_i$ be the coordinates in the system $\bar{x}$, where

$$\bar{x}_i = b_{ij} x_j, \quad b_{ik} b_{jk} = \delta_{ij}, \quad | b_{ij} | = 1$$