PERTURBATION THEORY AND NONPERTURBATIVE EFFECTS:
A HAPPY MARRIAGE?

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Perturbation expansions in renormalized quantum field theories are reformulated in a way that permits a straightforward handling of situations when in the conventional approach, i.e. in fixed renormalization scheme, these expansions are factorially divergent and even of asymptotically constant sign. The result takes the form of convergent (under certain circumstances) expansions in a set of functions $Z_k(a, \chi)$ of the couplant and the free parameter $\chi$ which specifies the procedure involved. The value of $\chi$ is shown to be correlated to the basic properties of nonperturbative effects as embodied in power corrections. Close connection of this procedure to Borel summation technique is demonstrated and its relation to conventional perturbation theory in fixed renormalization schemes elucidated.

1. Introduction

The problem of the convergence of perturbation expansions in quantum field theory is an old one, but with the advent of QCD it has acquired a new importance [1]. The necessity to rely in practice on only a few (in fact at most three [2]) lowest orders of perturbation theory moreover implies, in the situation when the relevant expansion parameter is by no means small like in QED, that the question of finding practical algorithms for summing these expansions is likely to be closely related to another problem plaguing the finite order approximations and namely the renormalization scheme (RS) ambiguity. Let me start by recalling that in QCD the perturbation expansions for physical quantities evaluated in conventional RS (like the MS, MOM etc.) are likely to be not only factorially divergent but even of asymptotically constant sign [3], i.e. like

$$\sum_{k=0}^{\infty} r_k x^{k+1}; r_k \rightarrow A(-\alpha)^k (\gamma k)! k^{\beta} x^{k+1} \text{ as } k \rightarrow \infty; \quad \alpha > 0, \gamma \leq 1, \quad (1.1)$$

where $A, \alpha, \beta, \gamma$ are the process and kinematics dependent variables. This makes a profound difference with respect to theories like QED or $\lambda \phi^4$ where similar expansions, though also factorially divergent, are oscillating, roughly like

$$\sum_{k=0}^{\infty} r_k x^{k+1}; r_k \rightarrow A(\alpha)^k (\gamma k)! k^{\beta} x^{k+1} \text{ as } k \rightarrow \infty; \quad \alpha > 0, \gamma \leq 1. \quad (1.2)$$

We understand the finiteness of the Borel sum of series like (1.2) as being due to subtle cancellations between large numbers of opposite signs whereas no such
intuitive picture exists for nonoscillating series (1.1). Indeed, it seems difficult to imagine any way of summing series (1.1) which would be reflected also in its truncated form, i.e., could be reasonably approximated by only a finite number of terms therein. It is sometimes claimed that for series with asymptotic behaviour (1.1–1.2) the “error” of its truncated form is given by the first neglected term and so one should truncate it once the highest calculated (and then discarded) term is the smallest. This is indeed true for the oscillating series (1.2) (assuming $\alpha = 1, \beta = 0, \gamma = 1$) summed according to Borel technique, but has otherwise no general validity. I would like to emphasize at the very beginning the obvious fact that there is in fact an infinite number of functions $F(x)$ which have (1.1) or (1.2) as their asymptotic expansions at $x = 0$. They may, moreover, differ wildly at $x > 0$ even for the oscillating series (1.2). For the nonoscillating series of the type (1.1) there is no “natural” candidate for $F(x)$ comparable to the Borel sum (or its modifications) at all. In general all the possible definitions of (1.1–1.2) differ by nonperturbative terms which behave like $\exp(-\gamma/x^\beta)$ at the origin. It is therefore only natural to look for such a definition of the sum of (1.1–1.2) that takes into account the most important part of the known nonperturbative contributions to $F(x)$. This can be done by specifying the full analytical properties of $F(x)$, but I shall argue that some progress can be achieved even by incorporating only the presently incomplete information on the relevant power corrections. In this paper I shall describe a procedure how to get finite and nontrivial results for physical quantities, which within the conventional perturbation theory in fixed renormalization scheme are given by divergent expansions in appropriate expansion parameters. The main idea of this procedure will be illustrated on a toy example of a power series with coefficients $r_k = k!$. Different points of view will then be considered in order to provide physical motivation for this way of summing divergent perturbation expansions. My interest in this field has been prompted by the idea of Stevenson [4] who showed how the renormalization group (RG) invariance of the theory can, under certain circumstances, lead to finite results even for highly divergent series. Contrary to him and some authors [5] I, however, do not think that this invariance, when applied to divergent series, implies by itself a unique sum if such a sum can be defined at all. The role of the renormalization procedure in the construction of nontrivial quantum field theories suggests [6] that this procedure cannot be regarded as purely perturbative in nature. It binds intimately together all aspects of the full theory and thus its perturbative and nonperturbative parts should somehow feel each other's presence. This is indeed the case in the present construction. The paper is organized as follows. In the next Section the nature of the problem is recalled, necessary notation introduced and the standard wisdom on how to handle divergent series briefly recalled. The basic idea of this paper is formulated in Section 3, followed in Section 4 by a detailed discussion of three different lines of reasoning leading to it. For the sake of simplicity and not to obscure the essence of the proposed method $c = 0$ is assumed in both of these sections (for definition of this variable see eq. (2.2) below). In Section 5 I shall then return to the real world with $c \neq 0$. Phenomenological implications of our method are briefly mentioned in Section 6. Summary and conclusions are reserved for the last section.