NOTE ON QUANTIZING CONSTRAINED SYSTEMS

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Some problems occurring in quantizing the constrained systems in which the elimination of non-physical variables is not unambiguous are illustrated by simple examples. The modification of the standard procedure of quantization is proposed in terms of path integral formalism.

1. INTRODUCTION

In many cases going from lagrangean $L$ to hamiltonian $H$ we find out that momenta $p = (p_1, p_2, \ldots p_n)$ are not independent. Then the relations of the type $A_a(p, q) = 0$, $a = 1, 2, \ldots, m < n$, $(q = (q_1, q_2, \ldots, q_n))$ will occur in the theory. The concise classical and quantum theory of such systems can be found e.g. in [1-3].

In this note we want to remark on the interesting feature of constrained systems. Namely, the presence of the constraints $A_a = 0$ results that some pairs of canonically conjugate variables are non-physical and they can be eliminated from the theory. But the elimination of non-physical variables is not unambiguous in general. This ambiguity does not represent any difficulty in the classical theory but in quantizing the theory we encounter the difficulties which (as far as we know) have not been discussed in the literature.

To illustrate our previous assertions let us consider the lagrangean proposed by Petráš [4] for a relativistic particle ($c = 1$)

$$L = \frac{m}{2} \frac{\dot{q}_i^2}{2} - \frac{1}{2} \left( m + \frac{M_0^2}{m} \right) + L_{\text{int}}(\dot{q}_i, q_i, t), \quad (i = 1, 2, 3)$$

where $q_i(t)$, $m(t)$ ($i = 1, 2, 3$) are coordinates, $\dot{q}_i = dq_i/dt$, $M_0$ is the rest mass of a particle, $L_{\text{int}}$ describes an interaction and repeated indices are summed over throughout the paper. In what follows we shall put $L_{\text{int}} = 0$. Since $L$ is independent of $dm/dt$ we have $p_m = 0$ ($p_m$ denotes the conjugate momentum to $m$) and hamiltonian is

$$H = \frac{p_i \dot{p}_i}{2m} + \frac{1}{2} \left( m + \frac{M_0^2}{m} \right).$$

The requirement $dp_m/dt = 0$ results in $\partial H/\partial m = 0$. It is straightforward to verify that the constraints $p_m = \partial H/\partial m = 0$ are consistent with the equations of motion and we have $m(t) = \pm \sqrt{(p_ip_i + M_0^2)}$. The interpretation of the solutions corresponding to $m(t) < 0$ can be the following. If we take into account the interaction with
an electromagnetic field then the classical trajectory of a particle with \( m(t) = -\sqrt{(p_i p_i + M_0^2)} \) and electric charge \(+e\) is the solution of the equations of motion with \( m(t) = +\sqrt{(p_i p_i + M_0^2)} \) and charge \(-e\). Moreover, one can easily see that a particle can move either in the domain \((p_i, q_i, p_m = 0, m \geq M_0)\) or in \((p_i, q_i, p_m = 0, m \leq -M_0)\) and there is no transition between these domains.

The difficulties which can emerge in quantizing the constrained systems are demonstrated in the next section.

II. QUANTIZING THE CONSTRAINED SYSTEMS

Let us assume

\[
\{H, A_a\} = u_{ab} A_b \quad \{A_a, A_b\} = v_{abc} A_c
\]

where \(\{,\}\) is the Poisson bracket of the quantities in question. As \(H\) and \(H' = H + \lambda_a A_a\) (\(\lambda\)'s are arbitrary functions of \(p, q, t\)) are entirely equivalent hamiltonians then the time-development of the physical quantities \(F\)'s cannot be influenced by the choice of \(\lambda\)'s, i.e. \(F\)'s have to satisfy

\[
\{F, A_a\} = z_{ab} A_b .
\]

It follows from the Eqs. (3, 4) that \(F\)'s are unambiguously determined in the certain domain \(F^*\) the dimension of which is \(2(n - m)\). So we can choose \(m\) additional constraints (gauge conditions) \(B_a = 0\). If \(\det M \neq 0\), where \(M_{ab} = \{A_a, B_b\}\), then the phase trajectory which starts in \(F^*\) continues in \(F^*\) too. The advantageous choice of \(B\)'s is (of course if this choice is possible) \(B_a = p_{n-m+a} = 0\) (or \(B_a = q_{n-m+a} = 0\)). Now the retarded Green function \(K\) of the considered system can be written in the form of continual integral

\[
K = \int \exp \left( \frac{i}{\hbar} S \right) \prod_{a=1}^{m} \delta(A_a) \delta(B_a) \left| \det M \right| (2\pi \hbar)^{-n+m} \prod_{k=1}^{n} dp_k(t) \ dq_k(t)
\]

where \(S = \int_{0}^{t} dt \ (p_k \dot{q}_k - H)\). The form of the right-hand side of the Eq. (5) is determined by the requirement which resides in the fact that after integration over variables \((p_{n-m+a}, q_{n-m+a}), a = 1, 2, \ldots, m\), we have to obtain

\[
K = \int \exp \left( \frac{i}{\hbar} S^* \right) \prod_{k=1}^{n-m} \frac{dp_k(t) \ dq_k(t)}{2\pi \hbar}
\]

where \(S^* = \int_{0}^{t} dt \ (p_k \dot{q}_k - H^*), k = 1, 2, \ldots, (n - m)\), i.e. our system became an ordinary Hamiltonian one after the elimination of non-physical variables. However, the Eq. (6) follows from the Eq. (5) provided that the constraints \(A_a = B_a = 0\) have unique solution for \((p_{n-m+a}, q_{n-m+a})\). If this assumption does not hold we have