Abstract. The concepts of [A,E,R(B)] and restricted [E,A,R(B)] invariance are introduced. The reachable subspace of a descriptor system is shown to be the supremal [A,E,R(B)]-invariant subspace contained in the least restricted [E,A,R(B)] subspace of $R^n$. Algorithms to compute the reachable subspace of a descriptor system $E\dot{x} = Ax + Bu$ in terms of $E,A$ and $B$ are given. A new proof of the feedback invariance of the reachable subspace is presented.

1. Introduction

We consider the linear, time-invariant descriptor system

$$D\Sigma: E\dot{x}(t) = Ax(t) + Bu(t)$$

(1.1)

where $E,A : C^n \rightarrow C^n$ and $B : C^m \rightarrow C^n$. We assume that $|\lambda E - A| \neq 0$, i.e., $D\Sigma$ is regular.

It is well known that there exists a basis $[v_i : i \in n]$ for the domain (where $n = [1,2, \ldots, n]$) and a basis $[w_i : i \in n]$ for the codomain of $E$ and $A$ such that in the new coordinates (1.1) decomposes into two subsystems (see[1])

$$D\Sigma WF_f : \dot{x}_1(t) = Jx_1(t) + B_1u(t)$$

(1.2a)

$$D\Sigma WF_{\infty} : \dot{x}_2(t) = x_2(t) + B_2u(t)$$

(1.2b)

where $x_1(t) \in C^{n_1}$ and $x_2(t) \in C^{n_2}$. $J$ and $N$ in (1.2) are Jordan form matrices and $N$ is nilpotent with index of nilpotency $\alpha_1$.

It was shown in [2] and [3] that the trajectory of (1.2) starting from an arbitrary initial condition may exhibit impulsive behavior. We shall call a point $x_0 = [x_{01}, x_{02}]'$ an admissible initial condition for (1.2) if there exists an $(\alpha_1 - 1)$ times continuously differentiable input $u_r(t) : [0,\infty) \rightarrow C^m$ such that the solution $x(t; 0, x_0, u_r(t))$ is continuously differentiable on $[0,T]$ for...
some $T > 0$. With some minor changes in the analyses of [3] and of [4], it was shown in [5] that $[x'_0, x''_0]'$ is an admissible condition for (1.2) if and only if $[x'_0, x''_0]' \in C^{n_1} \oplus C_\infty$ where $C_\infty = \Sigma_{i=0}^{\alpha_i-1} N_i R(B_2) + N(N)$ and $\oplus$ denotes direct sum.

Given a point $x_0 = [x'_0, x''_0]' \in C^{n_1+n_2}$, we say that $y_0 = [y'_0, y''_0]'$ is 
reachable from $x_0$ if there exists an $(\alpha_1 - 1)$ times continuously differentiable input $u_T(t)$ such that $x(t; 0, x_0, u_T(t)) = y_0$ for some $T > 0$. In case the origin is reachable from $x_0$, $x_0$ is said to be controllable. A slight modification of the analysis of [3,4] as carried out in [5] shows that any point in $R_F \oplus R_\infty$ is reachable from any other point in $R_F \oplus R_\infty$ where $R_F = \Sigma_{i=0}^{\alpha_i-1} J_i R(B_2)$ and $R_\infty = \Sigma_{i=0}^{\alpha_i-1} N_i R(B_2)$. It also follows from [2] and [5] that if $C_\infty = R_F$ and if $C_\infty$ is as above, then a point $[x'_0, x''_0]'$ is controllable if and only if $[x'_0, x''_0]' \in C_F \oplus C_\infty$. Consequently, $R_F \oplus R_\infty$ and $C_F \oplus C_\infty$ are called the reachable and controllable subspaces of (1.2). Equation (1.2) is said to be reachable (controllable) if $R_F \oplus R_\infty = C^{n_1+n_2}(C_F \oplus C_\infty = C^{n_1+n_2})$. This definition of reachability is equivalent to the controllability of [3], C-controllability of [4], and the absence of input decoupling zeros (finite or infinite) of $[sE-A B]$ in the sense of [6]. On the other hand, controllability, as defined above, is equivalent to modal controllability of [2].

This analysis of reachability and controllability, which is the prevalent approach used in the literature, has a main drawback in that it depends on the decomposition of $D\Sigma$ into two subsystems $D\Sigma F_F$ and $D\Sigma F_\infty$. A characterization of the reachable and the controllable subspaces in terms of $E, A$, and $B$ should be welcome for two main reasons. First of all, the decomposition has a computational cost associated with it. Secondly, decomposing the system into two subsystems (using the Weierstrass decomposition or any other one [3,7]) destroys one of the main advantages of the descriptor variable formulation. As it was pointed out in [6,8], the descriptor variable formulation is preferable to the state-space formulation, even when the latter exists, simply because it is "more natural" in the sense of displaying and preserving the information content of the dynamical equations and the physical significance of the variables. It is exactly this advantage which is being subverted by the decomposition of the system into two subsystems.

In Section 3 we circumvent these difficulties by showing that the reachability and the controllability of a descriptor system can be assessed and the reachable and the controllable subspaces can be constructed by using the original system matrices $E, A$, and $B$. The results of Section 3 depend on some geometric concepts which are introduced and discussed in Section 2. Section 4 will give an alternative proof of Cobb's result in [3] which states that the reachable subspace is feedback invariant.

In what follows, the superscript $-1$ on a linear operator will denote its pre-image and $\oplus$ will be used to denote the direct sum of subspaces and/or of linear operators.