Quasi-One-Dimensional Magnetohydrodynamic Flow with Heat Addition

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It is the purpose of this paper to present a method for discussing quasi-one-dimensional magnetohydrodynamic flows, subjected to a transverse magnetic field, with heat addition. When the pressure is used as an independent variable, a system of equations is obtained in which only derivatives of the entropy along a particle path appear [9]. Using known techniques [1], this system is written in characteristic form from which generalized Riemann invariants are obtained. Then limiting the analysis to small area variations and heat addition, the governing equations are linearized in the neighborhood of a known isentropic state and general solutions are obtained through the use of techniques derived by Paul Germain and the author [3, 4, 5]. The solutions obtained reduce to the solution for the corresponding problem in conventional gas dynamics in the limit of vanishing magnetic field.

The quasi-one-dimensional unsteady motion of an ideal, inviscid, perfectly conducting compressible fluid with heat addition, subjected to a transverse magnetic field, i.e., the induction \( \mathbf{B} = (0, 0, B) \), is governed by the system of equations [2]:

\[
\begin{align*}
P &= \exp \left[ \frac{(s - s^*)}{c_v} \right] q^\gamma, \\
q_t + u q_x + q u_x + q u \frac{A_x}{A} &= 0, \\
q u_t + q u u_x + P_x + \frac{B B_x}{\mu} &= 0, \\
B_t + u B_x + B u_x &= 0, \\
s_t + u s_x &= q,
\end{align*}
\]

where \( u, q, P, s, s^*, A, b^2 = B^2 / \mu, \gamma, c, q \) are, respectively, the particle velocity, density, pressure, specific entropy, specific entropy at some reference state, cross-sectional area, square of the Alfvén speed, permeability, ratio of specific heat at constant pressure \( c_p \) and at constant volume \( c_v \), local speed of sound and rate of entropy production (due to the heat addition). Partial derivatives are denoted by subscripts, and all dependent variables are functions of \( x \) and \( t \) alone save \( A \) which is considered time-independent. The characteristics of this system are:

\[
\frac{dx}{dt} = u, \quad u + \omega, \quad u - \omega
\]

where \( \omega = [b^2 + c^2]^1/2 \), the limiting case of a fast wave.

By the use of Equation (1), the derivatives of \( q \) may be replaced by derivatives of \( P \), and this gives the equation:

\[
P_t + u P_x + \gamma P u_x - P \left[ \frac{q}{c_v} - \gamma u \frac{A_x}{A} \right] = 0.
\]

2) Numbers in brackets refer to References, page 296.
To determine the characteristic form of the system of Equations (2'), (3), (4), and (5) let these be multiplied by \( v_1, v_2, v_3, \) and \( v_4 \), respectively. When the resulting equations are added, the following equation is obtained:

\[
\begin{align*}
v_1 P_t + [u v_1 + v_2] P_x + q v_1 u_t + [\gamma P v_1 + q u v_2 + B v_2] u_x + v_3 B_t \\
+ \left[ v_2 \frac{B}{\mu} + u v_2 \right] B_x + v_4 s_t + u v_4 s_x &= v_1 P \left[ \frac{q}{c_v} - \gamma u \frac{A_x}{A} \right] + v_4 q .
\end{align*}
\] (6)

The condition that (6) contain only derivatives in the characteristic direction is expressed by equality of the ratios:

\[
\frac{u v_1 + v_2}{v_1} = v_1 \gamma P + q u v_2 + v_3 B = \frac{v_2 B}{\mu} + u v_2 = \frac{u v_4}{v_4} = \frac{dx}{dt} = u + a .
\] (7)

The solutions are:

\[
\begin{align*}
v_2 &= a , \\
v_3 &= q \frac{a^2 - c^2}{B} , \\
v_4 &= \text{arbitrary} .
\end{align*}
\] (8)

The characteristic form of the basic system is obtained by substituting (8) into (6) for the three cases of interest, viz., \( a = \pm \omega, 0 \). This gives the following characteristic system:

\[
\begin{align*}
x_\beta - (u + \omega) t_\beta &= 0 , \\
x_\alpha - (u - \omega) t_\alpha &= 0 , \\
x_\xi - u t_\xi &= 0 ,
\end{align*}
\] (9)

\[
\begin{align*}
P_\beta + q \omega u_\beta + q (\omega^2 - c^2) \frac{B_\beta}{B} - P \left[ \frac{q}{c_v} - \gamma u \frac{A_x}{A} \right] &= 0 , \\
P_\alpha - q \omega u_\alpha + q (\omega^2 - c^2) \frac{B_\alpha}{B} - P \left[ \frac{q}{c_v} - \gamma u \frac{A_x}{A} \right] &= 0 ,
\end{align*}
\] (10)

\[
\begin{align*}
\frac{P_\xi}{\gamma P} - \frac{B_\xi}{B} - \left[ \frac{q}{\gamma c_v} - u \frac{A_x}{A} \right] &= 0 ,
\end{align*}
\] (11)

with characteristic parameters \((\alpha, \beta, \xi)\). Approximate solutions to the characteristic system may be obtained by finite-difference techniques.

To this point, the results apply to an arbitrary quasi-one-dimensional flow with heat addition. Now, let it be assumed that the cross-sectional area variations and heat addition are small and may be considered as small perturbations in an otherwise uniform base flow. Then from Equation (11), it follows that \( B/q \) is constant along each particle path of the base flow, which is a well-known consequence of the assumption of infinite electrical conductivity, i.e., the magnetic field is <<frozen>> into the fluid, and for a constant base flow, \( B/q \) is constant throughout the flow so that Equations (12) and (13) lead to:

\[
\begin{align*}
\frac{u}{2} + \frac{1}{2 \gamma} \int \omega \frac{dP}{P} &= \alpha , \\
- \frac{u}{2} + \frac{1}{2 \gamma} \int \omega \frac{dP}{P} &= \beta
\end{align*}
\]

where \((\alpha, \beta)\) may be considered generalizations of the usual Riemann invariants, cf., [6, 7, 8]. Thus with these assumptions, the basic system of equations is equivalent to the following system:

\[
\begin{align*}
\frac{\omega}{\gamma P} \left[ P_t + (u + \omega) P_x + [u_t + (u + \omega) u_x] - \frac{\omega}{\gamma} \left[ \frac{q}{c_v} - \gamma u \frac{A_x}{A} \right] = 0 ,
\end{align*}
\] (12)

\[
\begin{align*}
\frac{\omega}{\gamma P} \left[ P_t + (u - \omega) P_x - [u_t + (u - \omega) u_x] - \frac{\omega}{\gamma} \left[ \frac{q}{c_v} - \gamma u \frac{A_x}{A} \right] = 0 ,
\end{align*}
\] (13)

\[
\begin{align*}
s_t + u s_x &= q .
\end{align*}
\] (14)