Concerning Kustaanheimo-Stiefel's Regularization

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An isoenergetic transformation of the time followed by a conformal mapping from Cartesian to parabolic coordinates regularizes the singularity at the center of attraction for the two-dimensional problem of two bodies. Levi-Civita's technique has been extended to the three-dimensional case by KUSTAANHEIMO and STIEFEL. These authors solve in an explicit way the regularized equations of motion for the elliptic motion only, and they indicate how to present the solution in the other cases, the parabolic and hyperbolic motions. But their regularized equations of motion split into four linear homogeneous systems of the second order, each of which represents a harmonic oscillator. Hence, bypassing the usual discussion of the solutions according to the sign of the energy, we solve the reduced systems by means of the so-called universal functions recently introduced by STUMPF. In this manner we present the general solution of Kustaanheimo-Stiefel's regularized equations in terms valid for all Keplerian motions. It is quite interesting to observe that, when it is transferred back to the Cartesian coordinates, the general solution is identical in form to that presented by STUMPF.

This simple instance indicates clearly where the differences are between 'regularizations' and 'universal formulas' for the problem of two bodies. As a matter of fact, these methods are meant to answer questions of a different nature. Regularization is concerned with continuing analytically linear Keplerian motions beyond the singularity at the center of attraction; the 'universal formulas' purport to consider the set of all Keplerian motions as a function of the energy \( H \) and to give a uniform representation of that function in the neighborhood of \( H = 0 \) (for which the motions are either parabolic or linear parabolic). Often in the literature the two methods are contrasted. Actually, whether in the original Cartesian coordinates or in a system of regularizing coordinates, the Keplerian motions ought to be expressed in universal formulas. As we show here in the case of Kustaanheimo-Stiefel's mapping, it amounts to adopting the initial positions and velocities in the regularization phase space as the constants of motion rather than the usual orbital elements.

Let \( \mathbf{x}_i \) and \( \mathbf{X}_i \), where \( i = 1, 2, 3 \), be the coordinates and impulses in a 6-dimensional phase space; define the distance

\[
r = \sqrt{x_1^2 + x_2^2 + x_3^2}
\]

and the Hamiltonian function

\[
\mathcal{H} = \frac{1}{2} (X_1^2 + X_2^2 + X_3^2) - \frac{\mu}{r},
\]

where \( \mu \) is a strictly positive constant. The canonical system

\[
\dot{x}_i = \frac{\partial \mathcal{H}}{\partial X_i}, \quad \dot{X}_i = -\frac{\partial \mathcal{H}}{\partial x_i}, \quad i = 1, 2, 3,
\]

where the dot denotes differentiation with respect to the independent variable \( t \), admits the integral of energy

\[
\mathcal{H} = H.
\]

In consequence, for an arbitrarily fixed energy \( H \), define the Hamiltonian function

\[
\mathcal{K} = \frac{1}{2} r (X_1^2 + X_2^2 + X_3^2) - H r
\]
and generate the canonical system

\[ x'_i = \frac{\partial \mathcal{K}}{\partial X_i}, \quad X'_i = -\frac{\partial \mathcal{K}}{\partial x_i} \quad i = 1, 2, 3, \quad (4) \]

where the prime denotes differentiation with respect to the independent variable \( s \). Now the function \( \mathcal{K} \) thus defined has the following well known property: Those solutions of (1) along which (2) is satisfied are in 1–1 correspondence with the solutions of (4) along which the integral relation

\[ \mathcal{K} = \mu \quad (5) \]

is satisfied, provided the independent variables \( s \) and \( t \) are related by the quadrature

\[ t = \int r(s) \, ds. \quad (6) \]

2. Let \( \mu_\lambda \) and \( U_\lambda \), where \( \lambda = 1, 2, 3, 4 \) be the coordinates and impulses in an 8-dimensional phase space. Put

\[ x_1 = u_1^2 - u_2^2 - u_3^2 + u_4^2, \quad x_2 = 2 \left( u_1 u_2 - u_3 u_4 \right), \quad x_3 = 2 \left( u_1 u_3 + u_2 u_4 \right), \quad \text{and} \]

\[ U_1 = 2 \left( u_1 X_1 + u_2 X_2 + u_3 X_3 \right), \quad U_2 = 2 \left( -u_2 X_1 + u_1 X_2 + u_4 X_3 \right), \]

\[ U_3 = 2 \left( -u_3 X_1 - u_4 X_2 + u_1 X_3 \right), \quad U_4 = 2 \left( u_4 X_1 - u_3 X_2 + u_2 X_3 \right). \quad (8) \]

Let it be noted that

\[ \sum_\lambda u_\lambda^2 = r, \quad (9) \]

\[ \sum_\lambda u_\lambda U_\lambda = 2 \sum_i x_i X_i, \quad (10) \]

\[ \sum_\lambda U_\lambda^2 = 4 r \sum_i X_i^2. \quad (11) \]

On substituting (9) and (11) into (3) one obtains a function of the \( u_\lambda \) and \( U_\lambda \), which will be denoted by \( \mathcal{K}^* \), so that

\[ \mathcal{K}^* = \sum_\lambda \left( \frac{1}{8} U_\lambda^2 - H u_\lambda^2 \right). \quad (12) \]

Then generate the canonical system

\[ u'_\lambda = \frac{\partial \mathcal{K}^*}{\partial U_\lambda}, \quad U'_\lambda = -\frac{\partial \mathcal{K}^*}{\partial u_\lambda}, \quad \lambda = 1, 2, 3, 4. \quad (13) \]

Now the function \( \mathcal{K}^* \) has the following property [1]. For those solutions of (13) along which the relation

\[ u_4 U_1 - u_3 U_2 + u_2 U_3 - u_1 U_4 = 0 \quad (14) \]

is satisfied, the canonical system (4) is equivalent to the canonical system (13).

Kustaanheimo’s phase variables separate the eighth order system (13) into the 4 equations

\[ u_\lambda'' - \frac{1}{2} H u_\lambda = 0, \quad \lambda = 1, 2, 3, 4. \quad (15) \]

3. Let \( \rho \) be an arbitrary positive integer. The series

\[ C_\rho(s, \sigma) = \sum_{n \geq 0} \frac{(-1)^n}{(2n + \rho)!} \sigma^n s^{2n} \]

converges absolutely for all values of \( s \) and \( \sigma \); further it converges uniformly throughout any bounded domain of \( s \) and \( \sigma \).