Some Remarks on the Elastically Supported Membrane

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I. Introduction

The problem of an elastically supported membrane has recently received much attention in the literature, see for instance [1], [8]. In [5], [7], [9], gradient estimates for the first eigenfunction $u$ are obtained by an indirect use of Hopf's first [2] and second [3] principles on the function $|\text{grad } u|^2 + \lambda_1 u^2$. In this paper, we obtain additional results by applying these Hopf's principles to the functions $|\text{grad } u|^2 u^{-2}$ and $|\text{grad } v|^2 u^{-2}$ which have already been considered for harmonic functions by Payne and Philippin in [4]. In the latter case in which $v$ is the second eigenfunction of a free membrane we are led to a comparison between the eigenvalues corresponding to the eigenfunctions $u$ and $v$.

Throughout this paper the boundary $\partial D$ of the vibrating membrane $D$ is assumed to be a $C^{2+\varepsilon}$ bounded surface in $\mathbb{R}^N$, so that the Helmholtz equation holds on $\partial D$. However this assumption can be relaxed in most cases. We do not attempt here to determine the minimum smoothness requirement on $\partial D$.

II. Derivation of Maximum Principles

We consider the problem of an elastically supported membrane,

\begin{align*}
\Delta u + \lambda_1(x)u &= 0 \quad \text{in } D, \\
\frac{\partial u}{\partial n} + \alpha(s)u &= 0 \quad \text{on } \partial D.
\end{align*}

(2.1)

Here $\Delta$ is the Laplace operator, $\partial/\partial n$ denotes differentiation with respect to the outward normal direction of $\partial D$, and $\alpha(s)$ is a given nonnegative function defined on the boundary $\partial D$. $u$ is the first eigenfunction, and $\lambda_1(x)$ is the corresponding eigenvalue.

We define the function

$$
\Phi = \frac{u_x u_{xi} u^2}{u^2} = \frac{q^2}{u^2},
$$

(2.2)

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where \(u_u u_u\) is the square of the gradient of \(u\), also abbreviated \(q^2\). From the generalized maximum principle (see [6] pp. 72–73), we know that \(\Phi\) takes its maximum value on \(\partial D\). This result will now be sharpened and extended by means of the following:

**Lemma 1.** The function \(\Phi\) defined in (2.2) satisfies the following differential inequality:

\[
\Delta \Phi = \frac{2(\beta - 1)\Phi_{,k} u_{,k}}{u} \geq \frac{2\lambda_2^2(\alpha)}{N} + \frac{4\lambda_1(\alpha)}{N} (\beta + 1)\Phi

+ 2\Phi^2 \left\{ \frac{1}{N} (\beta + 1)^2 - \beta^2 \right\} \tag{2.3}'
\]

where \(\beta\) is an arbitrary real parameter, and \(N\) is the dimension of the considered euclidean space.

**Proof.** We denote partial differentiation by a comma followed by one or two indices, and adopt the summation convention on repeated indices. On differentiating (2.2), we have

\[
\Phi_{,k} = \frac{2u_{,ik} u_{,i}}{u^2} - \frac{2q^2 u_{,k}}{u^3} = \frac{2u_{,i}^2}{u^2} w_{ik}, \tag{2.4}
\]

with

\[
w_{ik} = u_{,ik} + \frac{\beta q^2}{u} \delta_{ik} - (1 + \beta) \frac{u_{,i} u_{,k}}{u}, \tag{2.5}
\]

where \(\beta\) is an arbitrary real parameter and \(\delta_{ik}\) is the Kronecker symbol. Differentiating again, we obtain

\[
\Delta \Phi = \frac{2u_{,ik} u_{,ik}}{u^2} - \frac{8u_{,ik} u_{,i} u_{,k}}{u^3} + \frac{6q^4}{u^4}. \tag{2.6}
\]

By using (2.5), (2.1), and (2.2), the quantity \(u_{,ik} u_{,ik}/u^2\) may now be expressed as follows:

\[
u_{,ik} u_{,ik} = \frac{w_{ik} w_{ik}}{u^2} = \left\{ (N - 1) \beta^2 + 1 \right\} \Phi^2 + 2\lambda_1(\alpha) \beta \Phi

+ 2(1 + \beta) \frac{u_{,ik} u_{,i} u_{,k}}{u^3}, \tag{2.7}
\]

where \(w_{ik} w_{ik}\) can be estimated in the following way:

\[
N w_{ik} w_{ik} \geq (w_{ii})^2 = \lambda_2^2(\alpha) u^2 + [\beta(N - 1) - 1]^2 \frac{q^4}{u^2}

- 2[\beta(N - 1) - 1] \lambda_1(\alpha) q^2. \tag{2.8}
\]