On the Use of Variational Methods in Steady Magneto-Gas Dynamics

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Introduction

Magneto-gas dynamics is that macroscopic theory of plasma which makes no explicit assumptions directed at distinguishing between electrons and ions [1, 2, 3]. The relevant starting equations are:

1) the Maxwell equations
\[ \text{curl } H = \frac{4 \pi}{c} j + \frac{1}{c} \frac{\partial D}{\partial t}, \quad \text{curl } E = -\frac{1}{c} \frac{\partial H}{\partial t}, \quad \text{div } D = 4 \pi \eta, \quad \text{div } B = 0. \]

2) the Euler equations
\[ \varrho^* \frac{dv}{dt} = -\nabla p + \frac{\mu}{c} \{ j \times H \} - \varrho^* \nabla \psi^* + \text{div } \Pi. \]

3) the equation of continuity
\[ \frac{\partial \varrho^*}{\partial t} + \text{div } (\varrho^* v) = 0. \]

4) the equation of state
\[ p \vec{V} = n \ R \ T. \]

5) the first law of thermodynamics
\[ \delta \varrho = dU^* + p \ dV^*. \]

As it is possible with ordinary gas dynamics, here too [4] we can derive a differential equation for the velocity potential \( \psi \) in the case of a quasi-neutral plasma provided that there is
1) no heat conduction
2) infinite electric conductivity
3) no friction
4) \( (H \ V) \ H = 0 \), [5].

This differential equation runs as follows:
\[ \frac{1}{a^*} \left\{ \frac{\partial^2 \psi}{\partial t^2} - \frac{1}{2} \frac{\partial}{\partial t} (\nabla \psi)^2 \right\} = \frac{1}{a^*} \left\{ \frac{\partial \psi}{\partial t} - \frac{1}{2} (\nabla \psi)^2 \right\}. \]

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2) Numbers in brackets refer to References, page 680.
For the two-dimensional steady case one obtains from it the potential equation

$$\varphi_{xx} (\varphi_x^2 - a^*2) + \varphi_{yy} (\varphi_y^2 - a^*2) + 2 \varphi_x \varphi_y \varphi_{xy} = 0.$$  \hspace{1cm} (1)

If the flow velocity \(v = \sqrt{\varphi_x^2 + \varphi_y^2}\) reaches the value of the wave velocity \(a^*\), (1) will change the type, that is to say, the equation turns from the elliptic type: \(\varphi_x^2 + \varphi_y^2 < a^*2\) via the parabolic one: \(\varphi_x^2 + \varphi_y^2 = a^*2\) to the hyperbolic type: \(\varphi_x^2 + \varphi_y^2 > a^*2\) [6, 7, 8].

From the mathematical point of view this would mean that different methods of solution must be employed for the different ranges, that is to say, with most methods the transition from the subcritical to the supercritical range is not possible [9, 10]. The only exceptions which are known are the variational methods; they permit a homogeneous representation of the solution for both ranges.

The report under discussion deals with the solution of (1) according to the variational method of Ritz [11]. The example used is the flow past a cylinder (direction of the magnetic field in direction of the cylinder axis).

The analogous problem of gas dynamics was worked out by Braun [10] with a formulation \(\varphi = \varphi_1 + \varphi_2\), \(\varphi_1\) being the incompressible flow. Since Braun concentrated on the linearized problem (terms higher than of first order in \(\varphi_2\) were neglected), his solution only applies to the subcritical range. Blank [12], like Braun, also split up the flow into incompressible basic flow with superimposed disturbance; for the disturbance only terms with \(\cos \psi\) were taken into account. As is shown by the calculations of Imai [13] and Imai and Lamla [14], both approximations are somewhat too rough, since on the one hand the term \(\cos 3\psi\) considerably influences the potential, on the other hand, however, the terms which are higher than of first order in \(\varphi_2\) become important near the sound velocity. Gisbert [15], when dealing with this problem, also considers the term with \(\cos 5\psi\). It becomes apparent that in gas dynamics the term \(\cos 3\psi\) must not be neglected, but that otherwise the term \(\cos 5\psi\) does not essentially influence the potential; therefore the term \(\cos 3\psi\) in \(\varphi_2\) is taken into account in the formulation \(\varphi = \varphi_1 + \varphi_2\) of this report, \(\varphi_1\) again denoting the potential of the incompressible flow.

**Method of Ritz and Setting Up of the Variational Integral**

The basic idea is the following one: the given differential equation (1) that is \(L(u) = 0\) is understood in region \(G\) as Euler's equation of a variational problem

$$\delta \Omega = 0\,,$$  \hspace{1cm} (2)

$$\Omega = \int F(x, y; u, u_x, u_y, \ldots) \, dx \, dy\,.$$

This variational problem is solved directly by setting up, according to Ritz, the solution function \(u(x, y)\) as series with indeterminate coefficients \(c_v^{(n)}\)

$$u_v(x, y) = \sum_{n=1}^{n} c_v^{(n)} u_v(x, y)$$  \hspace{1cm} (3)

\(u_v(x, y)\) form a complete orthogonal system in \(G\). If we set (3) in (2), we get after integration

$$\Omega_n = P(c_1^{(n)}, c_2^{(n)}, \ldots, c_n^{(n)})\,.$$

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