An Analysis of Two-dimensional Aligned-field Magnetogasdynamic Flows

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1. Introduction

The general magnetogasdynamic problem has usually proved to be too complex for a complete analytical solution to be obtained, but by considering certain classes of flows and certain types of fluids, the general equations can be reduced to simpler forms, and various techniques may then be applied to obtain a solution (see, for example, Goldsworthy [1], Cowley [2] and Apanasevich [3]). The aim of this paper is to provide a more detailed and thorough analytical background to some aspects of the magnetogasdynamic problem than has apparently hitherto been achieved.

An inviscid finitely-conducting fluid is assumed to move steadily and two-dimensionally. It is then found that the basic equations can be simplified to two sets of equations corresponding to two distinct regions of flow. It is shown that in regions where the electric charge density is non-zero, the magnetic field must be irrotational. In regions where the electric charge density is zero, the basic equations can be greatly simplified.

A great deal of attention has been given to infinitely conducting fluid flows in which the velocity and magnetic fields are everywhere aligned. By employing a complex variable technique this type of flow is analysed for a finitely conducting fluid. This technique can also be applied to other types of flow-fields.

2. Equations of Motion

The general equations governing the motion of an inviscid gas in steady motion will be taken in the form (see Ferraro and Plumptton [4])

\[ \nabla \cdot H = 0 , \]  
(1)

\[ \nabla \cdot E = \frac{q}{\epsilon} , \]  
(2)

\[ \nabla \wedge E = 0 , \]  
(3)

\[ \nabla \wedge H = j + q \mathbf{v} , \]  
(4)

\[ j = \sigma (E + \mu \mathbf{v} \wedge H) , \]  
(5)

\[ q (\mathbf{v} \cdot \nabla) \mathbf{v} = -\nabla p + \mu j \wedge H + q E , \]  
(6)

\[ \nabla \cdot q \mathbf{v} = 0 . \]  
(7)
Here, \( q \) denotes the electric charge density, \( j \) the density of the conduction electric current, \( \rho \) the gas density and \( \rho \) the gas pressure. The electrical conductivity of the field is denoted by \( \sigma \), the electric and magnetic fields by \( E \) and \( H \) respectively, and the velocity field by \( v \). Throughout, the permeability \( \mu \) and permittivity \( \varepsilon \) are taken to be those pertaining to free space. The units are M. K. S.

In addition to equations (1)–(7) there will be other equations connecting the heat flow \( Q \), temperature \( T \), entropy \( s \), and internal energy \( u \) together with previously defined quantities.

Firstly,

\[
T \, ds = du + \rho \, d \left( \frac{1}{q} \right),
\]

and secondly the equation of energy, assuming a steady flow, is

\[
\rho \, (v \cdot \nabla) \left\{ u + \frac{v \cdot v}{2} + \frac{\rho}{\varepsilon} \right\} = E \cdot j - \nabla \cdot Q.
\]

The entropy equation, being derivable from this latter equation together with the equation of motion (6) will not provide a further restriction.

For the purpose of this paper attention will be given solely to equations (1)–(7) and the above equations which will define to some extent the quantities \( Q, T, s \) and \( u \) will not be here investigated.

Assume the flow to be two-dimensional, so that \( v \) lies in a plane defined by the rectangular coordinates \( x, y \).

Elimination of \( j \) from equations (4) and (5) gives

\[
E = \frac{\nabla \wedge H}{\sigma} - \frac{q}{\sigma} v - \mu v \wedge H, \quad \sigma \neq 0, \tag{8}
\]

and the elimination of \( j \) and \( E \) from equations (4), (5), (6) yields

\[
\rho \, (v \cdot \nabla) v = -\nabla \rho + \mu (\nabla \wedge H) \wedge H - \frac{q^2}{\sigma} v, \tag{9}
\]

and

\[
2 \mu q v \wedge H = \frac{q}{\sigma} \nabla \wedge H, \quad \sigma \neq 0. \tag{10}
\]

Equations (3) and (8) yield the two equations

\[
\nabla \wedge \frac{q}{\sigma} v = 0, \tag{11}
\]

and

\[
\frac{\nabla \wedge (\nabla \wedge H)}{\sigma} = \mu \nabla \wedge (v \wedge H). \tag{12}
\]

Also equation (2) may be replaced by

\[
\frac{1}{\sigma} \nabla \cdot q v = -\frac{q}{\varepsilon}. \tag{13}
\]

Hence the basic equations are equivalent to equations (8)–(13) together with equations (1), (4) and (7).

For the class of flows where \( H \) is everywhere parallel to \( v \), \( H \) must be of the form \( K q v \), where \( K = K(x, y) \). Equations (1) and (7) imply that \( K \) is constant along