On an Integral Method for Backward Boundary Layers

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1. Introduction

In a recent paper, Goldstein [1] defined a backward boundary layer as one in which the fluid has been flowing along a solid wall over an infinite distance. In these cases if the coordinate $x$ is measured upstream along the boundary, the velocity $U_e(x)$ just outside the boundary layer will be negative and a favorable pressure gradient will be required to prevent separation. The similarity solution of Pohlhausen [2] (or see Goldstein [3], p. 143) which describes the steady two-dimensional flow along a wall in a converging passage is a flow of this type which has been studied extensively. This solution is unrealistic, however, because a potential-line-sink must be placed at the origin.

Most similarity solutions are limited in one way or another because it is not possible to impose an initial condition. Nevertheless, similarity solutions are valuable because they are correct in some limiting or asymptotic sense, and if Pohlhausen’s solution is interpreted in this way, it would be expected to be valid at a large distance from a finite slot through which fluid is being withdrawn. This assumes, of course, that the flow outside the boundary layer is potential and that no separation has occurred. When the fluid is close to the slot, the pressure distribution impressed on the boundary layer will not be that of a line-sink, but will depend on the detailed geometry of the container. The purpose of this paper is to develop a simple approximate method for integrating the boundary layer equations using the Kármán integral equation, when the flow is of the backward boundary layer type with an apparent singularity at the origin.

It is interesting that Pohlhausen used his similarity solution to illustrate the use of the Kármán-Pohlhausen integral technique in his pioneering paper [2]; the good agreement that he found probably convinced many readers of the merits of the approximate method, and we know now that it has many. It does not seem to be widely known that Dryden [4] discovered an error in Pohlhausen’s work, which after correction makes the results for the integral method imaginary. Dryden extended the range of applicability of Pohlhausen’s method by using a fifth-order polynomial and in this way was able to apply the modified method to Pohlhausen’s similarity solution with good accuracy, but only after some cumbersome calculations. Since the similarity solution is strictly valid at infinitely large distances, it is not clear how the integral method can be used when the integration must be started at $x = \infty$ and
$U_e(x)$ has the form

$$U_e = - \frac{1}{x} + \frac{b_1}{x^2} + \frac{b_2}{x^3} + \ldots$$  \hspace{1cm} (1.1)

Here a method is developed using a third order polynomial which is 'tailored' to handle flows for which $U_e(x)$ is given by (1.1). $U_e$'s of this form will arise whenever there is a two-dimensional potential flow toward a slot or opening, and the resulting backward boundary layer flows are probably the most common type to be observed.

In Sections 2 and 3, the problem is formulated mathematically and a differential equation is obtained for the shape factor. To eliminate the difficulty of starting the numerical integration at $x = \infty$, $U_e(x)$ is chosen as a new independent variable (in place of $x$) in Section 4 and a numerical procedure is outlined. In Section 5, the method is applied to the flow toward a two-dimensional slot, and the results are found to be in excellent agreement with those obtained by Thwaites' method [5]. The method presented here is generalized to other backward boundary layers in Section 6, and a comparison is made with some of the calculated Falkner-Skan solutions. Stagnation point flows are also soluble by this method and the boundary layer flow on a flat plate set perpendicular to a uniform stream is considered. In all cases there is very good agreement with known exact solutions and Thwaites' method. Although the method is best suited to flows where there is a favorable pressure gradient, it could be used to start the integration in a region of favorable pressure gradient, and then a switch-over to a more conventional method could be made when the pressure gradient is less favorable.

2. Mathematical Formulation

Introduce a coordinate system where $\bar{x}$ is the distance measured along the bounding surface from some origin and $\bar{y}$ is the distance normal to the surface. Dimensional variables will be denoted by bars. The space coordinates, velocity components, and pressure are nondimensionalized as follows:

$$x = \frac{\bar{x}}{L}, \quad Y = \frac{\bar{y} \sqrt{Re}}{L}, \quad u = \frac{\bar{u}}{U_0}, \quad v = \frac{\bar{v} \sqrt{Re}}{U_0}, \quad \phi = \frac{\bar{p}}{\rho U_0^2}. \quad (2.1)$$

Here $L$ and $U_0$ are a length and velocity which are characteristic of the flow, $\rho$ is the constant fluid density, and $Re = \rho U_0 L/\mu$ is the Reynolds number where $\mu$ is the fixed fluid viscosity. When the Reynolds number is large, Prandtl's boundary layer equations are applicable and they may be written in the nondimensional form

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial Y} = 0, \quad (2.2)$$

and

$$\frac{u}{\partial x} \frac{\partial u}{\partial x} + v \frac{\partial u}{\partial Y} \frac{\partial u}{\partial Y} = \frac{U_e \partial U_e}{\partial x} + \frac{\partial^2 u}{\partial Y^2}. \quad (2.3)$$

$U_e(x)$ is the nondimensional $x$-component of velocity just outside the boundary layer and it is assumed known from the potential flow solution or by experiment. In either case, it can be written in the form (1.1) for $x \to \infty$.

1) In this equation it is assumed that $U_e$ and $x$ have been made non-dimensional in such a way that the coefficient of $-x^{-1}$ is unity. This can be done without loss of generality.