On the Exact Solutions of Vibration Problems of Rectangular Orthotropic Plates

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Introduction

The analysis of free vibration of rectangular orthotropic plates has been considered by HUFFINGTON and HOPPMANN [1]. They obtained the frequency equations for rectangular plates with two opposite sides simply-supported and the other two sides having any other support. HEARON [2] used the characteristic beam functions satisfying the boundary conditions and obtained closed formulas for the frequencies by applying the Rayleigh method. More recently the free vibration of rectangular and circular orthotropic plates was considered by the author [3] by applying GALERKIN's method. In this paper certain formulas are obtained from which the exact frequencies of orthotropic plates can be readily calculated from those of the corresponding isotropic plates. Rectangular plates are considered with two opposite sides simply-supported and one of the other two sides elastically restrained against rotation with any other support on the remaining side. The use of the formulas is illustrated under case (1) by taking a numerical example.

Analysis

The motion of a transversely vibrating plate of homogeneous orthotropic material according to classical small deflection theory is governed by [1]

\[
D_x \frac{\partial^4 w}{\partial x^4} + 2H \frac{\partial^4 w}{\partial x^2 \partial y^2} + D_y \frac{\partial^4 w}{\partial y^4} - \rho_1 h \frac{\partial^2 w}{\partial t^2} = 0,
\]

where \(D_x, D_y\) are the flexural rigidities, \(H\) is the torsional rigidity, \(\rho_1\) is the circular frequency, \(w\) is the deflection, \(h\) is the thickness and \(\rho_1\) is the mass density of the orthotropic plate. Taking the axes of coordinates along the sides of the plate with dimensions \(a_1, b_1\) (Figure 1) and introducing the non-dimensional quantities defined by

\[
x = a_1 \xi; \quad y = b_1 \eta,
\]

Equation (1) takes the form

\[
\left( \frac{h}{m_1} \right)^4 \frac{\partial^4 w}{\partial \xi^4} + 2 \alpha_1 \left( \frac{h}{m_1} \right)^2 \frac{\partial^4 w}{\partial \xi^2 \partial \eta^2} + \frac{\partial^4 w}{\partial \eta^4} - \pi^4 \frac{h}{m_1} \lambda_1^2 \omega = 0,
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\]
where
\[ k_1 = m_1 \left( \frac{b_1}{a_1} \right) \alpha_2 ; \quad \alpha_2 = \left( \frac{D_x}{D_y} \right)^{1/4} ; \quad \alpha_1 = \frac{H}{(D_x D_y)^{1/4}} ; \quad \lambda_1^2 = \frac{\varrho_1 k_1 b_1^4 b_4^4}{k_1^2 \pi^4 D_y}. \] (4)

The sides of the plate now become \( \xi = 0, \eta = 1 \) and \( \eta = 0, \eta = 1 \). Taking the sides \( \xi = 0, \xi = 1 \) to be simply-supported and following the method of Lévy [4], \( w \) can be taken as
\[ w = f(\eta) \sin m_1 \pi \xi. \] (3)

Using Equation (5) in Equation (3) gives
\[ \frac{d^2 f}{d \eta^2} - 2 \pi^2 \alpha_1^2 \alpha_2 \frac{d^2 f}{d \eta^2} + \pi^4 \alpha_1^4 \left( 1 - \lambda_1^2 \right) f = 0. \] (6)

The solution of Equation (6) can be written as
\[ f(\eta) = C_1 \cos \theta_1 \eta + C_2 \sin \theta_1 \eta + C_3 \cos \psi_1 \eta + C_4 \sin \psi_1 \eta \] (7)
where
\[ \phi_1^2 = \pi^2 \alpha_1^2 \left[ \alpha_1 + (\alpha_2^2 + \lambda_1^2 - 1)^{1/2} \right], \quad \psi_1^2 = \pi^2 \alpha_1^2 \left[ -\alpha_1 + (\alpha_2^2 + \lambda_1^2 - 1)^{1/2} \right]. \] (8)

The frequencies of isotropic and orthotropic plates are now connected by certain formulas for the following cases.

Case 1: For a plate clamped along \( \eta = 0 \), and elastically restrained against rotation on \( \eta = 1 \), the boundary conditions are
\[ f(\eta = 0) = f'(\eta = 0) = 0 ; \quad f''(\eta = 1) = -\frac{4 s_1 b_1}{D_y} f'(\eta = 1) \] (9a)

where \( s_1 \) is the stiffness per unit length of the elastic restraining medium at the edge of the plate, or the moment required to rotate a unit length of the medium through one fourth of the radian (Lundquist and Stowell [5]). Letting \( t_1 = 4 s_1 b_1 / D_y \) in Equation (10) and using Equations (9) and (10) in Equation (7), the characteristic frequency equation is obtained as
\[ (\phi_1^2 + \psi_1^2) \left( \psi_1 \sin \phi_1 \cos \psi_1 - \phi_1 \cosh \phi_1 \sin \psi_1 \right) - t_1 \phi_1 \psi_1 \times \left[ (\cos \psi_1 - \cosh \phi_1)^2 + \frac{1}{\phi_1 \psi_1} (\phi_1 \sin \psi_1 - \psi_1 \sin \phi_1) (\psi_1 \sin \psi_1 + \phi_1 \sin \phi_1) \right] = 0. \] (11)

The frequency equation of the corresponding isotropic plate is
\[ (\phi^2 + \psi^2) \left( \psi \sin \phi \cos \psi - \phi \cosh \phi \sin \psi \right) - t \phi \psi \times \left[ (\cos \psi - \cosh \phi)^2 + \frac{1}{\phi \psi} (\phi \sin \psi - \psi \sin \phi) (\psi \sin \psi + \phi \sin \phi) \right] = 0, \] (12)
where
\[ \phi^2 = \pi^2 \kappa^2 \left( 1 + \lambda \right), \quad \psi^2 = \pi^2 \kappa^2 \left( -1 + \lambda \right), \quad t = \frac{4 s b}{D}, \] (13)
with
\[ \kappa = \frac{m b}{a}, \quad \lambda^2 = \frac{\varrho b \kappa^2 b^4}{\pi^4 k^4 D}. \] (14)

\( D \) is the flexural rigidity of the isotropic plate with sides \( a \) and \( b \) and circular frequency \( \varrho \). Hence the conditions for the frequency equation (11) to reduce to that of the corresponding isotropic plate and vice-versa are
\[ \phi_1 = \phi, \quad \psi_1 = \psi, \quad t_1 = t, \] (15)
which give the following useful affine formulas.
\[ \kappa_1^2 \alpha_1 = \kappa^2, \] (16a)