Reciprocal Relations in Non-Steady One-Dimensional Gasdynamics

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1. Introduction

The invariance of the basic equations of gasdynamics and magnetogasdynamics under suitable transformations has been the subject of a number of papers recently. In particular, one may cite the related work of A. A. Nikol’skii [1], E. D. Tomilov [2] and V. A. Rykov [3] on non-steady flows, based on a paper by L. V. Ovsianikov [4] concerning group properties of differential equations. Further, employing considerations of invariance, P. Smith [5] has obtained a substitution principle for a restricted class of non-steady flows of a Prim gas.

Reciprocal relations in two-dimensional steady gasdynamics and magnetogasdynamics have been developed in a series of papers by G. Power and P. Smith [6], G. Power and A. Tunbridge [7], G. Power and D. Walker [8, 9]. These are concerned with establishing the invariance of the governing equations with reciprocal-type mappings and thereby linking a known solution with a four-parameter class of solutions, each with its own equation of state.

In this paper, reciprocal relations are constructed for non-steady one-dimensional inviscid gasdynamics, and more generally for the one-dimensional flow of an infinitely conducting gas in the presence of a transverse magnetic field. An immediate extension may also be made to cover ($\varepsilon + 1$)-dimensional spherically symmetric flows.

Essentially, the work deals with the inverse method of constructing new classes of solutions to the governing equations of gasdynamics (magnetogasdynamics) using the concept of reciprocity. Hence we will not be here concerned with applying the results to initial or boundary value problems. For a discussion on the role of inverse methods in gasdynamics, reference may be made to P. F. Neményi [10].

2. The Reciprocal Relations

The basic equations of inviscid one-dimensional non-steady gasdynamics neglecting heat conduction and heat radiation are

\[ \rho_t + (\rho u)_x = 0, \]  
\[ \rho (u_t + uu_x) + p_x = 0, \]  
\[ s_t + uu_s = 0, \]
together with an equation of state
\[ f(p, q, s) = 0, \quad \left( \frac{\partial p}{\partial q} \right)_s > 0 \quad (q \neq 0), \]
where \( x, t, u, p, q \) and \( s \) denote respectively linear co-ordinate, time, velocity, pressure, density and specific entropy.

Equations (1), (2) imply the existence of \( \psi(x, t), \psi'(x, t) \) such that
\[ d\psi = q \, dx - q \, u \, dt, \]
\[ a_1 \, dt' = q \, u \, dx - (p + q \, u^2 + a_2) \, dt, \quad a_1 \neq 0, \]
where \( a_1, a_2 \) are constants.

Introducing new variables \( u', q', x' \) defined by
\[ q' = a_1 \left( \frac{p}{p + a_2} \right), \]
\[ q' = \frac{a_3 \, q \left( \frac{p}{p + a_2} \right)}{(p + q \, u^2 + a_2)}, \quad a_3 \neq 0, \]
\[ x' = x, \]
it is seen that
\[ q' \, dx' - q' \, u' \, dt' = a_3 \left( q \, dx - q \, u \, dt \right), \]
implying that
\[ q' \, + (q' \, u')_x' = 0. \]
In (8) above, \( a_3 \) denotes an arbitrary non-zero constant.

Since
\[ \frac{\partial}{\partial x} = \frac{\partial}{\partial x'} + \frac{q}{a_1} \frac{\partial}{\partial t'}, \quad \frac{\partial}{\partial t} = - \left( \frac{p + q \, u^2 + a_2}{a_1} \right) \frac{\partial}{\partial t'}, \]
(2) transforms into
\[ q' \left( u'_x + u' \, u'_x \right) + \left[ \frac{a_3^2 \, a_3}{(p + a_2)^2} \right] p_x' = 0 \]
on using (7), (8). Hence, defining
\[ p' = a_4 - \frac{a_3^2 \, a_3}{(p + a_2)}, \]
where \( a_4 \) is an arbitrary constant, (11) becomes
\[ q' \left( u'_x + u' \, u'_x \right) + p'_x = 0 \]
so that (2) is preserved in the \( x', t' \)-space.

Finally, (3) yields
\[ s_t + u \, s_x = - \left( \frac{p + a_2}{a_1} \right) \left[ s'_x + u' \, s'_x \right] = 0, \]
so that, assuming \( p + a_2 \neq 0, \)
\[ s'_t + u' \, s'_x = 0, \]
where
\[ s' = \Phi(s) \]
is some function of the entropy.