ON THE TENSOR PRODUCT
OF SUPERSINGLETON REPRESENTATIONS OF \(osp(1, 2n)\)

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The tensor product of two supersingleton representations \(\sigma_n\) of the Lie superalgebra \(osp(1, 2n)\) is studied for \(n \geq 2\). The main results are as follows: (a) anticommutators and commutators of the odd generators in \(\sigma_n \otimes \sigma_n\) form a skew-symmetric representation of the Lie algebra \(u(n, n)\); (b) simple explicit form of all irreducible components of \(\sigma_n \otimes \sigma_n\), which are labelled by a single parameter \(J = 0, 1, \ldots\), has been found. Each of them is a \(*\)-representation of \(osp(1, 2n)\) for which assertion (a) is valid. The dimension of its vacuum subspace equals \(J + n - 1\), i.e., the nondegenerate vacuum occurs for \(J = 0\) only. Basic property of this family of irreducible \(*\)-representations of \(osp(1, 2n)\) are analogous to those of massless representations of \(osp(1, 4)\).

1. INTRODUCTION

During the last decade considerable effort has been made in studying infinite-dimensional representations of the Lie superalgebra \(osp(1, 4)\) \([1-4]\). This superalgebra is a supersymmetric extension of \(sp(4, \mathbb{R}) \sim so(3, 2)\), i.e., of the Lie algebra of space-time symmetries of the de-Sitter universe; when passing (by contraction) to the flat space-time, the Wess-Zumino superalgebra arises from \(osp(1, 4)\). Complete classification of \(*\)-representations of \(osp(1, 4)\) was given in terms of highest-weight representations \(D(E, j)\) of \(so(3, 2)\) and, on the other hand, various concrete realizations of these representations were found.

From the point of view of possible physical applications it seems worthwhile to generalize what we have learned about \(osp(1, 4)\) for both values of indices, i.e., for the Lie superalgebras \(osp(N, 4), N = 2, 3, \ldots\) and \(osp(1, 2n), n = 3, 4, \ldots\). Our main concern in this paper is the second case while the first one, which represents the supersymmetric extension of the de-Sitter space-time symmetry combined with the internal symmetry \(so(N)\), will be dealt with elsewhere \([5]\). The reason why representations of \(osp(1, 2n)\) are of physical interest consists first of all in their connection to the para-Bose quantization (parastatistics) \([6]\). It is well known that the odd generators of any \(*\)-representation of \(osp(1, 2n)\) fulfil the para-Bose quantization rules and vice versa. Although no experimentally observed particle obeys parastatistics, there are indications that parastatistics may be suitable for singletons — the

*) Dedicated to Academician Václav Votruba on the occasion of his eightieth birthday.
hithetical unobservable constituents of massless (and possibly also massive) particles that are supposed to be of more fundamental character than quarks [7].

In the present paper the results of our recent paper [8] are generalized for \( osp(1, 2n) \) with arbitrary \( n \geq 3 \). We consider the tensor product of two supersingleton (or oscillator) representations \( \sigma_n \) of \( osp(1, 2n) \) — a straightforward generalization of the supersingleton representation \( \sigma_2 \) of \( osp(1, 4) \) whose restriction to \( so(3, 2) \) equals direct sum of the Dirac singletons \( D(\frac{1}{2}, 0) \) and \( D(1, \frac{1}{2}) \). After recalling in Sec. 2 some facts about \( osp(1, 2n) \) and \( \sigma_n \), we show in the next section that the representation \( \sigma_n \otimes \sigma_n \) has the following remarkable property: by adding to \( (\sigma_n \otimes \sigma_n) \) the commutators of the odd generators of \( osp(1, 2n) \), a skew-symmetric representation of \( u(n, n) \) arises.

The second problem solved in this paper concerns irreducible components of \( \sigma_n \otimes \sigma_n \) (Sec. 4). We succeeded in finding all of them in a very simple explicit form. In this way a class of irreducible \(*\)-representations of \( osp(1, 2n) \) with both nondegenerate and degenerate vacuum was obtained. Clearly, each of these representations has the property of Sec. 3. In the concluding section some implications and generalizations of the presented results are discussed. They include the problem of reducing the components of \( \sigma_n \otimes \sigma_n \) with respect to \( sp(2n, \mathbb{R}) \), connection of the results of Sec. 3 to the Ohnuki-Kamefuchi trilinear relations etc.

2. PRELIMINARIES

1. The \( osp(1, 2n) \) is a real Lie superalgebra that is generated by \( 2n \) odd elements \( y_{\alpha} \), \( \alpha = \pm 1, \ldots, \pm n \). Their anticommutators determine \( n(2n + 1) \) even elements

\[
x_{\alpha \beta} := \frac{1}{2} \{ y_{\alpha}, y_{\beta} \} = x_{\beta \alpha}
\]

obeying the following commutation relations:

\[
[x_{\alpha \beta}, y_{\gamma}] = g_{\alpha \gamma} y_{\beta} + g_{\beta \gamma} y_{\alpha}, \quad g_{\alpha \beta} := \varepsilon_{\alpha} \delta_{\alpha + \beta}, \quad \varepsilon_{\alpha} := \text{sgn}(\alpha),
\]

2. Let \( \mathcal{H} \) be a Hilbert space, \( \mathcal{D} \) a dense subspace of \( \mathcal{H} \) and denote by \( \text{End} \mathcal{D} \) the set of linear operators on \( \mathcal{H} \) that are defined on \( \mathcal{D} \) and leave it invariant. The map \( \pi: osp(1, 2n) \rightarrow \text{End} \mathcal{D} \) is called a \(*\)-representation if

(i) \( \mathcal{D} \) is \( Z_2 \)-graded, \( \mathcal{D} = \mathcal{D}_0 \oplus \mathcal{D}_1 \), and \( \pi \) is a (graded) representation of \( osp(1, 2n) \) on \( \mathcal{D} \) (see § I.1 of Ref. [9]);

(ii) all the operators \( iX_{\alpha \beta} \) and \( \exp(-i\pi/4) Y_{\alpha} \), where \( X_{\alpha \beta} := \pi(x_{\alpha \beta}), Y_{\alpha} := \pi(y_{\alpha}) \), are symmetric.

3. For a given \(*\)-representation \( \pi \) of \( osp(1, 2n) \) introduce

\[
A_{\alpha} := 2^{-1/2}(Y_{\alpha} - iY_{-\alpha}),
\]

\[
B_{\alpha \beta} := \frac{1}{2}(A_{\alpha}, A_{\beta}) = \frac{1}{2}(X_{\alpha \beta} - X_{-\alpha - \beta}) - \frac{i}{2}(X_{-\alpha \beta} + X_{-\alpha - \beta}).
\]