Asymptotic Behavior of Solutions to Certain Nonlinear Schrödinger-Hartree Equations*

R. T. Glassey
Department of Mathematics, Indiana University, Bloomington, Indiana 47401, USA

Abstract. The asymptotic behavior of solutions to the Cauchy problem for the equation

\[ i\psi_t = \frac{1}{2} \Delta \psi - v(\psi) \psi, \quad v = r^{-1} * |\psi|^2, \]

and for systems of similar form, is studied. It is shown that the norms

\[ \|\psi(t)\|_{L^2(|x| \leq R)}^2 + \|\partial_t \psi(t)\|_{L^2(|x| \leq R)}^2 \]

are integrable in time for any fixed \( R > 0 \), from which it follows that

\[ \lim_{t \to \infty} \|\psi(t)\|_{L^2(|x| \leq R)} = 0. \]

Nevertheless, it is established that an \( L^2 \)-scattering theory is impossible.

Introduction

We consider classical solutions to the Cauchy problem for the equations

\[ i\psi_t = \frac{1}{2} \Delta \psi - v(\psi) \psi \quad (x \in \mathbb{R}^3, \ t > 0) \tag{1} \]

\[ v(\psi) = r^{-1} * |\psi|^2 = \int_{\mathbb{R}^3} |x - y|^{-1} |\psi(y, t)|^2 dy \quad (r = |x|) \]

and

\[ i\psi_j/\partial t = \frac{1}{2} \psi_j - \sum_{k=1}^{N} (\psi_j \psi_k - \psi_k \psi_j) \quad (j = 1, 2, \ldots, N) \tag{2} \]

where

\[ \psi_j = r^{-1} * \psi_j \psi_k, \quad \psi_k = \psi_{kk} = r^{-1} * |\psi_k|^2. \]

Equations (1), (2) are Coulomb-free versions of the time-dependent Hartree and Hartree-Fock equations. In [2] we have treated the existence question for

* Research supported in part by National Science Foundation Grant GP37630
Equations (1), (2) with coulomb terms present, and have shown that global solutions exist with the quantity [for (2)]

$$\sum_{j=1}^{N} \left\{ \| \psi_j(t) \|^2 + \| V\psi_j(t) \|^2 \right\}$$

remaining uniformly bounded. The notation here is

$$\| \psi(t) \|_2 = \left( \int_{\mathbb{R}^3} |\psi(x, t)|^2 \, dx \right)^{1/2},$$

etc. A similar result is valid for solutions to (1).

In this paper we shall obtain the following results: Let \( \psi \) be a solution of (1) with finite energy norm (as above). Then for every fixed \( R > 0 \) we have

$$\int_{0}^{\infty} \int_{|x| \leq R} (|\psi|^2 + |V\psi|^2) \, dx \, dt < \infty$$

from which we conclude that

$$\lim_{t \to \infty} \int_{|x| \leq R} |\psi|^2 \, dx = 0.$$ 

For spherically symmetric solutions \( \psi_j \) of (2) with finite energy, the same results are valid for each \( \psi_j, 1 \leq j \leq N \). However, we also show that an \( L^2 \)-scattering theory for non-trivial solutions of (1) is impossible. It is plausible that solutions to (1) do decay uniformly to zero as \( t \to \infty \), but if so, the rate of decay cannot be fast enough to insure the existence of asymptotic free states.

The desired estimates follow from an identity obtained essentially through use of the multiplier \( \partial \psi / \partial r \) [for (1)]. The resulting estimate is the exact analogue of Morawetz' estimate [5]. During the preparation of this work, the author learned that this multiplier was found independently and, in fact, first, by Lin. In his thesis [4], Lin studies the asymptotic behavior of solutions to equations of the form

$$iu_t = \Delta u - h(x)q(|u|^2)u$$

and shows that

$$\| u(t) \|_{\infty} = O(t^{-3/2}) \quad \text{as} \quad t \to \infty$$

under certain conditions on \( h \) and \( q \). In addition, decay of the “local energy norm” is established, and a scattering theory is developed.

The reason for the restriction to spherically symmetric solutions of (2) involves the use of a radial derivative as a multiplier; this will be evident from the proof.

Although we dealt in [2] only with generalized solutions, it is easy to see that in the absence of coulomb terms [i.e. for (1), (2)], solutions will be smooth if the data is. By induction on \( k \) we can show that the norms \( \sum_{|\alpha| \leq k} \| D^\alpha \psi \|_2 \) are finite for all \( t \geq 0 \). For \( k \leq 2 \), this was done in [2]. For higher values of \( k \), we write the potential \( v \) [in the case of Equation (1)] as

$$v = \int_{\mathbb{R}^3} |z|^{-1} |\psi(x - z, t)|^2 \, dz$$

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1 The restrictions involving \( h \) can be removed, e.g. \( h \equiv 1 \) is admissible (private communication from Prof. Walter Strauss)