THE PRESENT APPROACH TO THE STUDY OF THE LEAST-SQUARES METHOD

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In Gauss' original model $H = P^{-1}$, where $P$ is the diagonal matrix of weights $\{P\} = p_i > 0$ and, as regards the matrix $A$, $R(A) = m \leq N$, where $R(.)$ represents the rank of the matrix in the parentheses. Under these assumptions the LSM determines the linear mapping of the vector $x$ into vector $\hat{\Theta}$, which is an estimate of vector $\Theta$ and which satisfies condition

\[(1) \quad (x - A\hat{\Theta})' P(x - A\hat{\Theta}) = \min \{(x - At)' P(x - At) : t \in \mathbb{R}^m\},\]

where $\mathbb{R}^m$ is the $m$-dimensional real vector space of parameters.

We now interpret Eq. (1) from the point of view of various metric spaces in dependence on the properties of matrix $P$.

1. Assume $P = I$; in the $N$-dimensional Euclidean space $(\mathbb{R}^N, \rho_e) \{\rho_e(x, y) = \sqrt{(x - y)'(x - y)} \}, x, y \in \mathbb{R}^N\}$ Eq. (1) is then equal to the distance between point $x$, which characterizes the set of measured data in $\mathbb{R}^N$, and that element of subspace $M(A)$, generated by the columns of matrix $A$, which can be expressed as a linear combination of the columns of matrix $A$ with coefficients equal to the components of the vector $t$. Therefore, the LSM estimate $\hat{\Theta}$, if $P = I$, minimizes the Euclidean norm between $x$ (the set of measured data) and the $A$-image of the LSM estimator $\hat{\Theta}$. This is then a question of projecting the vector of the measured data onto subspace $M(A)$.

2. If $P + I$, the LSM replaces Euclidean space with the metric space $(\mathbb{R}^N, \rho_p) \{\rho_p(x, y) = \sqrt{(x - y)' P(x - y)} \}$. In the case of non-diagonal but regular matrices of the weighting coefficients $H$, Aitken [1] considered the metric $\rho_H(x, y) = [(x - y)' H^{-1}(x - y)]^{1/2}$, which is Mahalanobis's metric [2]. In this case the projector $A(A'A)^{-1} A'$ is replaced by the projector $A(A'PA)^{-1} A'P$ or $A(A'H^{-1}A)^{-1} A'H^{-1}$, which again is the projector onto subspace $M(A)$ in space $(\mathbb{R}^N, \rho_H)$ or $(\mathbb{R}^N, \rho_P)$, respectively [6].

The justification for the mentioned method of determining the linear estimator $\hat{\Theta}$ of the vector $\Theta$ is implied by the statistical properties of the LSM mapping of $\Theta$: $\mathbb{R}^N \to \mathbb{R}^m$.

3. STATISTICAL PROPERTIES OF THE LSM AND NORMALITY

As regards the mapping $\hat{\Theta}: \mathbb{R}^N \to \mathbb{R}^m$, $\hat{\Theta}(x) = (A'H^{-1}A)^{-1} A'H^{-1}x$, which has property (1), it is true that

\[(2) \quad \forall\{\Theta \in \mathbb{R}^m\} E(\hat{\Theta}) = \Theta \& \forall\{c \in \mathbb{R}^m\} \forall\{\hat{\Theta}(.) \neq \hat{\Theta}(.) : \hat{\Theta}(.)\}
\]

is linear and unbiased \[\mathcal{D}(c' \hat{\Theta})^* \leq \mathcal{D}(c' \Theta);\]

$\mathcal{D}(.)$ represents the dispersion of the random variable in the parentheses. Property (2) represents the efficiency of the estimator $\hat{\Theta}$.

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