CONTRIBUTION OF THE FUNCTIONAL FORMULATION OF THE PROBLEM OF TURBULENCE TO THE PROCESS OF ITS ALGEBRAIZATION

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Summary: The progressive development of the theory of turbulence is based on the formalism of characteristic functionals. Drawing on an incomplete analogy between the Hopf equation for these functionals and the equation for the quantized boson field, a system of fundamental principles of the theory of turbulence is proposed. This represents a set of statements which can be used in an attempt to classify the theory in the sense of its algebraization and, therefore, also its axiomatization.

1. INTRODUCTION

The present formulation of the general problem of turbulence leads to a search for the probability distribution in the phase space of the turbulent flow — in the functional space of all possible non-divergent velocity fields, satisfying the equations of hydrodynamics and the boundary conditions. A complete statistical description of the turbulent velocity field will be achieved once we know the space-time characteristic functional of this field which, as proved by Hopf, can be obtained by solving a certain differential equations in terms of variational derivatives [1]. The fundamental problem of statistical hydrodynamics, formulated in terms characteristic functionals, whether space-time or spatial, is linear regardless of the fact that the dynamics of fluids belong to the field of non-linear mechanics.

The solving of the equations, derived by Hopf, are associated with mathematical difficulties typical of solving differential equations in terms of variational derivatives. The most effective method of solving them is the use of the so-called continuous integral. In physics this integral was first used by Feynman in 1948 [2, 3]. The apparatus of continuous integrals was introduced into the theory of turbulence by Rosen in 1960 [4]. Briefly speaking, this involves the integrals of functionals integrated with respect to some “generalized measure” in functional space.

The functional formulation of the problem of turbulence, apart from the most general aspect of the statistical dynamics of turbulence, also indicates the direction in which one can proceed in the process of its algebraization. A part of the algebraization of the theory of turbulence and, generally, of any discipline in which mathematics are involved to a considerable degree, is an attempt at establishing a theory, founded on several principles — statements, with the aid of which the axioms appropriate to the field may be formulated. The significance of the functional formulation of the problem of turbulence will be enhanced by further statements in the development of the quantum theory of the field, with which it has a common mathematical apparatus. This is essentially already inferred by the fact that in both theories systems of fields interacting with one another are being investigated — non-linear systems with a theoretically infinite number of degrees of freedom.

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2. COMPLETE STATISTICAL DESCRIPTION OF THE TURBULENT VELOCITY FIELD

The problem of turbulence, expressed in terms of characteristic functionals, is just a differently formulated problem of the distribution of probability in phase space of turbulent flow. If $\theta(x, t) = (\theta_1(x, t), \ldots, \theta_N(x, t))$ is a continuous function, identically equal to zero outside a certain limited space-time region, otherwise, however, quite arbitrary, the characteristic space-time functional of $N$-dimensional random velocity field $u(x) = (u_1(x), \ldots, u_N(x))$, of the vectorial argument $\theta(x, t)$, is defined by the mean value of the expression $\exp \{i \int \sum_{k=1}^N \theta_k(x, t) u_k(x, t) \, dx \, dt\}$. In this notation the round parentheses indicate the integral of the scalar product of functions in which the integration is carried out in the space-time region, filled with fluid, as a whole. The value of the characteristic functional $\Phi[\theta(x, t)] = \exp \{i(\theta \cdot u)\}$ at “points” $\sum_j \delta_j(x - x_j) \delta(t - t_j)$ of the space of functions $\theta(x, t)$ represents the characteristic function of the probability distribution for the values of the velocity fields selected points $x_j, t_j$ of space-time. We can also say that the characteristic functional $\Phi[\theta(x, t)] = \phi[\theta_1(x, t), \ldots, \theta_N(x, t)]$ of an $N$-dimensional random velocity field is the Fourier image of the $N$-dimensional probability density $p_{x_1,\ldots,x_N}(\theta_1, \ldots, \theta_N)$ of the values of $N$ random quantities $\theta_1, \ldots, \theta_N$ at points $x_1, \ldots, x_N: \phi_{x_1,\ldots,x_N}(\theta_1, \ldots, \theta_N) = \int \exp (i \sum_{k=1}^N \theta_k u_k) p_{x_1,\ldots,x_N}(\theta_1, \ldots, \theta_N) \, d\theta_1 \ldots \ldots, d\theta_N$ [3].

There exists a mutually unique relation between the characteristic function $\Phi[\theta(x, t)]$ and the moments or semi-variants of the random field $u(x)$ [3, p. 198]. This fact is employed in introducing the algebra of characteristic functionals as an algebra of observable quantities. The characteristic functional of the velocity field represents the most compact form of information on the problem of turbulence. Introducing it is equivalent to defining an infinite set of all possible moments of this field, occurring in the analytical description of turbulence which is expressed by an infinite system of Keleer-Friedman equations [3]*). It should be added that the well-known properties of correlation functions of pulsation of the hydrodynamic field are implied by the relations between the characteristic functional and the moments, as well as semi-invariants of the field $u(x)$.

*) We are able to prove that the functional formulation of the problem of turbulence will develop into a compactly expressed form of record of these equations in the appropriate formalism e.g., by substituting into Hopf’s equation for the functional $\Phi[\Theta(x, t)]$ all the terms of the series

$$\Phi = 1 + \sum_{n=1}^{\infty} \Phi_n$$

instead of $\Phi$. Here $\Phi_n$ is a homogeneous functional of the $n$-th degree which reads

$$\Phi_n = \int \Theta_j(x_1), \ldots, \Theta_j(x_n) \, dx_1, \ldots, dx_n.$$