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Remarks on Eigenvalues and Eigenvectors of Hermitian Matrices, Berry Phase, Adiabatic Connections and Quantum Hall Effect*

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Abstract. The singularities of the stratification of the space of the Hermitian matrices according to the multiplicities of the eigenvalues are described as an informal complexification of the previous study of the space of the real symmetric matrices. The degeneration of the spectral sequence associated to this stratification provides some strange combinatorial identities. The eigenvector bundles over the manifold of the Hermitian matrices with simple spectra are equiped with the natural connections, describing also the adiabatic approximation to the oscillations of the linear systems defined by the slowly varying skew Hermitian matrices. The curvature of this connection is singular at the codimension 3 variety of the Hermitian matrices having multiple eigenvalues. The resulting jumps of the integrals of the curvature form at the crossings of this variety by the moving surface of integration are responsible for the quantum Hall effect.

§1. Introduction

The space of Hermitian matrices is stratified according to the multiplicities of the eigenvalues. The subvariety formed by those matrices which have multiple eigenvalues has codimension *three* in the space of Hermitian matrices. The singular points of this subvariety correspond to the matrices having more multiple eigenvalues and higher multiplicities. They form the strata of codimension at least 6 in the space of Hermitian matrices. The simplest properties of this stratification have been described in [1]. In the present paper some new properties of this stratification and some problems and conjectures related to it are described.

The geometry of the space of the Hermitian matrices having no multiple eigenvalues is related to the adiabatic connections on the eigenvector bundles (known in physics under the name of the Berry phase), while the Hermitian matrices having multiple eigenvalues are responsible for the quantum Hall effect. The author is grateful to S.P. Novikov, who has explained to him the relation of the paper [1] to the quantum Hall effect, and is also grateful to A. Vainstein, M. Shapiro, B. Shapiro, A. Givental, V. Vassiliev, and A.I. Neistadt for their useful discussions.

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§2. The natural stratification of the space of Hermitian matrices

The strata are labeled by the vectors of the multiplicities of the ordered eigenvalues $\lambda_1 < \lambda_2 < \dots$. Our goal is the study of the singularities of the closures of these strata, of their adjacencies to each other, of the behavior of the natural fibrations of the eigenvectors over these strata and of the natural connections on these fibrations at the boundaries of the strata.

All the results might be formulated for the Lie algebra of the skew-Hermitian matrices, but I formulate them in the Hermitian case to preserve the parallelism with the stratification of the space of the real symmetric matrices.

It is technically convenient to consider the matrices whose traces vanish. To avoid the noncompactness, it suffices to consider the stratification of the sphere of those Hermitian matrices having trace zero, whose norms are equal to one.

Denote by $\mathcal{F}_0 \subset \mathcal{F}_1 \subset \mathcal{F}_2 \subset \cdots$ the filtration of the sphere by the closed subvarieties $\mathcal{F}_p$ of the Hermitian matrices having at most $p+1$ different eigenvalues (and having trace zero and norm one).

This filtration generates a spectral sequence, convergent to the (co)homologies of the sphere.

PROPOSITION. *The spectral sequence of the filtration $\{\mathcal{F}_p\}$ degenerates at the second term.*

The proof [2] of this proposition, found by M. Shapiro and A. Vainstein (to whom I had communicated this proposition as a conjecture) will be published elsewhere.

EXAMPLE. Consider the Hermitian matrices of order $n = 4$. The corresponding sphere has the dimension $n^2 - 2 = 14$. I shall denote the strata like $(\lambda_1 < \lambda_2 = \lambda_3 < \lambda_4)$ by the symbols like $(12|34)$. Our stratification of the sphere consists of $2^{n-1} - 1 = 7$ strata (corresponding to the vertices of a cube of dimension $n-1 = 3$). The adjacencies of these strata (and their real codimensions in S^{14}) are shown in Fig. 1.

Every stratum of $\mathcal{F}_p \setminus \mathcal{F}_{p-1}$ is fibered over an open simplex of dimension p (formed by the eigenvalues). Each fiber consists of the Hermitian matrices having the given eigenvalues with given multiplicities. These fibers are also shown in Fig. 1. They are the complex flag manifolds. Indeed, an Hermitian matrix is unambiguously defined by its eigenvalues and by their eigenspaces, which are pairwise Hermitian orthogonal. The set of the eigenspaces V_i (ordered by the order of the eigenvalues λ_i on the real axis) is bijectively defined by the flag of their sums $W_i = V_i + V_{i+1} + \cdots$.

For instance, the manifold of the complete flags in $\mathbf{C}^4$ is fibered over $\mathbf{CP}^3$ with a fiber which is fibered over $\mathbf{CP}^2$ with fiber $\mathbf{CP}^1$. Hence I denote the variety