RAY AMPLITUDES IN A THREE-DIMENSIONAL INHOMOGENEOUS MEDIUM

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Summary: A system of five ordinary differential equations of the first order to compute the geometrical spreading of the wave front of seismic body waves in a three-dimensional medium is suggested.

The methods of computing rays in general inhomogeneous media are well known. The rays are usually determined by solving a system of ordinary differential equations of the first order. The system consists of 3 equations in the two-dimensional medium and of 5 equations in a three-dimensional medium.

To compute the amplitudes of seismic body waves along rays is more complicated [1]. Difficulties are caused, in the first place, by the computing of the geometrical spreading of the wave front. The geometrical spreading of the wave front is closely related to the Jacobian of the transformation from Cartesian to ray coordinates, the absolute value of which is usually denoted by \( J \) [1, 2]. A number of methods can be used to compute \( J \). One of these is founded on the direct numerical measurements of the cross section of an elementary ray tube [1]. In the case of a three-dimensional velocity distribution, however, this method requires a minimum of three rays, close to another, to be computed. For this reason, methods, which enable the function \( J \) to be computed directly along a specified ray without auxiliary computations of other rays, are rather important. The function \( J \) may be expressed, e.g., with the aid of certain auxiliary quantities which can be determined by solving a system of ordinary differential equations of the first order, just like the ray. If the velocity depends on all three coordinates, the ray tracing system must be supplemented by ten other equations if the function \( J \) is to be determined. The complete system of ordinary differential equations then consists of 15 equations [1].

The large number of equations in this system considerably restricts its practical applications. Therefore, an attention has been devoted to decreasing the number of equations in the system. For example, Chen and Ludwig [3] proposed a system of ordinary differential equations, consisting only of 12 equations. However, decreasing the number of equations by 3 has made them considerably more complicated.

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A completely new approach to computing the function $J$ has been suggested by Popov and Pšeněk [2]. This approach is founded in particular on the choice of a suitable new coordinate system, which moves along the ray being investigated. In this way they were able to decrease the number of equations by two, so that their system consists of 13 equations. Thus, the number of equations in their system is one more than in [3], however, the equations appear to be simpler. Apart from this, their system also has other advantages which are described in [2].

The system of equations, proposed in this paper, consists only of 10 equations, 5 for the ray and 5 for the function $J$. In deriving it, the approach, proposed in [2], was adopted in full. In [2] one can also acquaint himself with the detailed introduction of individual quantities. The ray tracing system, consisting of 5 equations, is well known [1, 2], and we shall not describe it again. We shall only deal with the equations for determining the function $J$.

Consider an arbitrary medium, in which the velocity of propagation of elastic waves, $v = v(x, y, z)$, is a continuous function of the coordinates, together with its derivatives up to the second order. Consider any ray in the medium and select a length parameter $s$ on it with a specified initial point $s = s_0$. We know that in the ray theory an important role is played by the ray torsion $T(s)$. We introduce the angle $\theta(s)$ as follows

$$\theta(s) = \theta(s_0) + \int_{s_0}^{s} T(s) \, ds.$$  

Using the function $\theta(s)$ we can introduce a new system of coordinates $(s, q_1, q_2)$, referred to the specified ray. The first coordinate is $s$, the length parameter along the ray. At a given point of the ray, $s$, we construct a plane perpendicular to the ray, $S_\perp$, and on it two mutually perpendicular unit vectors $e_1$ and $e_2$, with their origin on the ray, by means of the relations

$$e_1 = n \cos \theta - b \sin \theta, \quad e_2 = n \sin \theta + b \cos \theta.$$  

Here $n$ and $b$ are the unit vectors of the normal and the binormal to the ray. The position of any point $M$, located on plane $S_\perp$, may then be expressed as

$$r(M) = r(s_M) + q_1 \, e_1(s_M) + q_2 \, e_2(s_M).$$  

Here $r(M)$ is the radius-vector of point $M$, $s = s_M$ is the length parameter along the specified ray, corresponding to the point of intersection of the plane $S_\perp$ with the ray, and $r(s_M)$ is the radius-vector corresponding to the said point of intersection. The unit vectors $t$, $e_1$ and $e_2$ ($t$ being the unit vector tangential to the ray, pointing in the direction of propagation of the wave) form a right-handed system of vectors.

The system of 8 ordinary differential equations of the first order, which can be used to determine the function $J$, was derived in [2]. The system reads

$$\begin{align*}
\frac{dQ_{11}}{ds} &= v_{11}, \\
\frac{dP_{11}}{ds} &= -v^{-2}(v_{11}Q_{11} + v_{12}Q_{21}), \\
\frac{dQ_{12}}{ds} &= v_{12}, \\
\frac{dP_{12}}{ds} &= -v^{-2}(v_{11}Q_{12} + v_{12}Q_{22}), \\
\frac{dQ_{21}}{ds} &= v_{21}, \\
\frac{dP_{21}}{ds} &= -v^{-2}(v_{21}Q_{11} + v_{22}Q_{21}), \\
\frac{dQ_{22}}{ds} &= v_{22}, \\
\frac{dP_{22}}{ds} &= -v^{-2}(v_{21}Q_{12} + v_{22}Q_{22}),
\end{align*}$$  

where

$$v_{ij} = \frac{\partial^2 v}{\partial q_i \partial q_j}.$$  

The quantities $v$ and $v_{ij}$ are being considered for $q_1 = q_2 = 0$, i.e. at the point of intersection of the plane $S_\perp$ with the ray. The function $J$ is then determined by the equation

$$J = |Q_{11}Q_{22} - Q_{12}Q_{21}|.$$  

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