SINGLE-LAYER DENSITY AS FUNCTION OF STOKES' CONSTANTS

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Summary: A relation between Stokes' constants (harmonic coefficients) \( J_n^{(k)} \), \( S_n^{(k)} \) has been derived in the development for the external geopotential and the coefficients in the development for the single-layer density distributed over the surface of the external ellipsoid, the external equipotential surface, as well as the smoothed physical surface of the actual Earth. Terms of the 2nd order, \((f_2^{(0)})^2\), were taken into account, terms of the 3rd and lower orders were neglected.

1. STOKES' CONSTANTS AND THE DENSITY OF AN EQUIVALENT SINGLE LAYER ON EXTERNAL SURFACES

Stokes' constants will be considered in the form of dimensionless coefficients:

\[
J_n^{(k)} = \frac{\delta_k (n - k)!}{M a_0 (n + k)!} \int \int \int \delta_0^m P_n^{(k)}(\sin \Phi) \frac{\cos kA}{\sin kA} \; d\tau = \\
= \frac{\delta_k (n - k)!}{G M a_0 (n + k)!} \int \int \varphi_S^m P_n^{(k)}(\sin \Phi) \frac{\cos kA}{\sin kA} \; dS ,
\]

\[
\delta_k = \begin{cases} 
1 & \text{if } k = 0 \\
\frac{1}{2} & \text{if } k \neq 0 
\end{cases}.
\]

\( M \) is the mass of the Earth, \( \delta \) the density in the volume element \( d\tau \) with geocentric co-ordinates \((\vartheta, \Phi, A)\), \( \varphi \) the equivalent single-layer density on the surface element \( dS \) with co-ordinates \((\vartheta_S, \Phi, A)\), and \( a_0 \) the length factor. The surface on which the single layer is going to be considered will be defined by the development

\[
\varphi_S = R_0 \left[ 1 + A_0^{(0)} + \sum_{n=2}^{N} \sum_{k=0}^{n} \left( A_n^{(k)} \cos kA + B_n^{(k)} \sin kA \right) P_n^{(k)}(\sin \Phi) \right] .
\]

It can be represented, e.g., by an equipotential surface with potential \( W_0 \); in this case \( R_0 = GM/W_0 [10] \). Its surface element \( dS \) is

\[
dS = R_0^2 \left[ 1 + s_0^{(0)} + \sum_{n=2}^{N} \sum_{k=0}^{n} \left( s_n^{(k)} \cos kA + t_n^{(k)} \sin kA \right) P_n^{(k)}(\sin \Phi) \right] \cos \Phi \; dA \; d\Phi ;
\]

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the expressions for \( s^{(0)}_n \), \( s^{(k)}_n \) and \( t^{(k)}_n \) are in [11]*). The single-layer density can be expressed by the development

\[
\psi = \Phi_0 \left[ 1 + \varphi^{(0)}_n + \sum_{n=2}^{N} \sum_{k=0}^{n} \left( \psi^{(k)}_n \cos kA + \psi^{(k)}_n \sin kA \right) P^{(k)}_n(\sin \Phi) \right],
\]

\[\Phi_0 = \frac{GM}{(4\pi R_0^3)} .\]

If one substitutes Eqs (2), (3) and (4) into (1), on neglecting the 3rd order terms one arrives at

\[
\begin{align*}
J^{(k)}(n) &= \frac{\delta(n-k)!}{4\pi(n+k)!} \left( \frac{R_0}{a_0} \right)^n \int_{-\pi/2}^{\pi/2} \int_{-\pi/2}^{\pi/2} P^{(k)}_n(\sin \Phi) \cos kA \left( (1 + \varphi^{(0)}_n)(1 + nA^{(0)}_n) \right) \\
&\quad \cdot (1 + s^{(0)}_n) + \frac{1}{6}n(n-1)(A^{(0)}_n)^2 + \left[ \varphi^{(0)}_2(1 + nA^{(0)}_0 + s^{(0)}_0) + nA^{(0)}_2(1 + \varphi^{(0)}_0 + s^{(0)}_0) + nA^{(0)}_0A^{(0)}_2 + s^{(0)}_2(1 + \varphi^{(0)}_0 + nA^{(0)}_0) \right].
\end{align*}
\]

With a view to the orthogonality of spherical functions Eq. (5) yields

if \( n = 0, k = 0 \):

\[
J^{(0)}_0 = (1 + \varphi^{(0)}_0)(1 + s^{(0)}_0) + \frac{1}{2} \varphi^{(0)}_2 s^{(0)}_2 ;
\]

if \( n = 2, k = 0 \):

\[
J^{(0)}_2 = \left( \frac{R_0}{a_0} \right)^2 \left[ \frac{1}{3} \varphi^{(0)}_4(1 + 2A^{(0)}_0 + s^{(0)}_0 + \frac{4}{3}A^{(0)}_2 + \frac{2}{3}s^{(0)}_2) + \frac{2}{3}A^{(0)}_4(1 + \varphi^{(0)}_0 + A^{(0)}_0 + s^{(0)}_0 + \frac{1}{3}A^{(0)}_2 + \frac{1}{3}s^{(0)}_2) + \frac{1}{3}A^{(0)}_6(1 + \varphi^{(0)}_0 + 2A^{(0)}_0) \right] ;
\]

if \( n = 4, k = 0 \):

\[
J^{(0)}_4 = \left( \frac{R_0}{a_0} \right)^4 \left( \frac{1}{15} \varphi^{(0)}_8(4A^{(0)}_0 + s^{(0)}_0) + \frac{2}{3}A^{(0)}_8(4A^{(0)}_0 + s^{(0)}_0) + 4A^{(0)}_4s^{(0)}_2 + 6(A^{(0)}_2)^2 \right) ;
\]

if \( n = 3; 5, 6, \ldots, N; k = 0 \):

\[
J^{(0)}_n = (2n + 1)^{-1} \left( \frac{R_0}{a_0} \right)^n \left( \varphi^{(0)}_n + nA^{(0)}_n + s^{(0)}_n \right) ;
\]

*) The symbol \( S^{(k)}_n \) in [11] is not suitable since it biases with the symbol of Stokes' constant with \( \sin kA \); that is why \( s \) and \( t \) are used here instead of \( S \) and \( T \).