On Local Görtler Instability

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1. Introduction

The instability of boundary-layer flows over concave walls to perturbations in the form of longitudinal vortices was first demonstrated theoretically by Görtler [1] in 1940. Ever since detailed studies of the transition from laminar to turbulent flow revealed the necessary involvement of three-dimensional disturbances, Görtler's analysis, descriptive as it is of a three-dimensional instability, has held an important place in the development of ideas concerning the nature of laminar boundary-layer instability.

Although the predictions of the theory have been confirmed under experimental conditions approximating those assumed in the formulation [2–4] many of its implications have not been as fully explored because certain of the theoretical assumptions limit its applicability to an as yet undetermined degree. One such assumption is that the wall, of infinite streamwise extent, have a constant curvature everywhere along its length. In practice, wall curvature normally is present over only
a limited extent. Clearly, for a wall with a given curvature over a limited extent of its length, reducing the extent likewise should reduce the tendency of the flow toward instability. Thus, use of the theory in a local sense becomes increasingly questionable as the extent of wall curvature diminishes.

Instances abound where one would like to use the Görtler theory in a local sense. One recalls that it is not wall curvature in itself that is important so much as it is the presence of streamline curvature, the curvature being concave in the direction of increasing stream velocity. All of the myriad instances where this is the case locally might be subject in some sense to a flow instability of the Görtler type. Görtler and his colleagues have attempted to extend the theory to some of these cases, notably to plane stagnation-point flow [5] and to flow perturbed initially by Tollmien-Schlichting waves [6].

The present paper also is directed to the question of applying the Görtler theory locally. Returning to the original problem of flow over a curved wall [1], we specify that the wall have concave curvature over only a limited streamwise extent, being plane elsewhere. By a suitable approximation, we derive a simple correction to the Görtler critical factor for instability that brings into evidence the factor’s dependence on the extent of wall curvature. The report concludes with a speculative example outlining how local Görtler instability might play a role as an intermediary in the transition to turbulence of the flow behind a fixed roughness element.

2. Analysis

2.1. Navier-Stokes Equations

Consider the flow of an incompressible viscous fluid over a wall arbitrarily curved in the streamwise direction. The equations governing the flow are the Navier-Stokes equations; written in invariant vector form (with the body force neglected), they are [7]

\[
\frac{\partial \mathbf{v}}{\partial t} - \mathbf{v} \times \mathbf{\omega} = - \text{grad} \left( \frac{\rho}{2} + \frac{1}{2} \mathbf{v} \cdot \mathbf{v} \right) + \mathbf{v} \left( \text{grad div} \mathbf{v} - \text{curl} \mathbf{\omega} \right),
\]

\[
\mathbf{\omega} = \text{curl} \mathbf{v}, \quad \text{div} \mathbf{v} = 0.
\]

In an orthogonal curvilinear coordinate system \((x_1, x_2, x_3)\), the components of the velocity vector \(\mathbf{v}\) are, respectively \(u_1, u_2, u_3\); \(\mathbf{\omega}\) is the vorticity vector, \(\rho\) the pressure, \(\varrho\) the density, \(\nu\) the kinematic viscosity, \(t\) the time. Let \(x_1\) be the streamwise coordinate measured as distance along the wall, \(x_2\) distance normal to the wall and \(x_3\) the spanwise coordinate normal to \(x_1, x_2\). The elements of length at \((x_1, x_2, x_3)\) are \(h_1 \, dx_1, h_2 \, dx_2, h_3 \, dx_3\), so that

\[
ds^2 = h_1^2 \, dx_1^2 + h_2^2 \, dx_2^2 + h_3^2 \, dx_3^2. \tag{2}
\]

A simple geometric construction yields (cf. Fig. 1)

\[
h_1 = \frac{r_0 - x_2}{r_0}, \quad h_2 = 1, \quad h_3 = 1 \tag{3}
\]