Differential renormalization, the action principle and renormalization group calculations

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Abstract. General prescriptions of differential renormalization are presented. It is shown that renormalization group functions are straightforwardly expressed through some constants that naturally arise within this approach. The status of the action principle in the framework of differential renormalization is discussed.

1 Introduction

Differential renormalization [1,2] was invented as an alternative renormalization scheme useful for calculations strictly in four dimensions [3,4]. The basic idea of this renormalization is to represent products of propagators in coordinate space through derivatives of sufficient order acting on locally integrable functions. Here the product is over the lines of a given graph $G_{\Gamma}$, $\pi_{\pm}(l)$ are respectively beginning and the end of a line $l$,

$$H(x_{1},\ldots,x_{N}) = \prod \Gamma I G_{\pi_{\pm}(l)} - x_{\pi_{\pm}(l)},$$

through derivatives of sufficient order acting on locally integrable functions. Here the product is over the lines of a given graph $\Gamma$, $\pi_{\pm}(l)$ are respectively beginning and the end of a line $l$,

$$G_{\Gamma}(x) = P_{\Gamma}(\partial/\partial x, m_{\Gamma}) \frac{m_{\Gamma}}{4\pi^{2}x^{2}} K_{1}(m_{\Gamma}\sqrt{x^{2}})$$

is a propagator, with $P_{\Gamma}$ polynomial and $K_{1}$ a modified Bessel function. This procedure explicitly characterizes the $R$-operation (i.e. renormalization at diagrammatic level) as an extension of the functional $H_{\Gamma}(x_{1},\ldots,x_{n})$ from the subspace of test functions which vanish in a vicinity of points where the coordinates $x_{i}$ coincide to the whole space $\mathcal{D}(R^{4})$.

The first step within the initial version of differential renormalization [1,2] is to reduce the problem to the case of diagrams depending on one coordinate difference. To do this at low orders of perturbation theory, it suffices to exploit certain manipulations based on the Leibniz rule. At higher orders, the only way of performing such a reduction is to integrate over all coordinate differences except one. However it is then possible to run into infrared problems since this 'naïve' integration generally induces infrared divergences. In [6] the original version of differential renormalization was supplied with simple prescriptions which enabled infrared troubles to be avoided so that differentially renormalized expressions could be found with no more difficulty than determining the corresponding counterterms in dimensional renormalization. It was also shown that in writing down differentially renormalized quantities it is very useful to apply calculational experience based on dimensional regularization.

The second step [1, 2] is performed with prescriptions of the following type:

$$\frac{1}{x^{4}} \rightarrow -\frac{1}{4} \Box \ln \mu^{2}x^{2}$$

$$\frac{\ln \mu^{2}x^{2}}{x^{4}} \rightarrow -\frac{1}{8} \Box \ln \mu^{2}x^{2} + 2 \ln \mu^{2}x^{2}$$

$$\frac{1}{x^{6}} \rightarrow -\frac{1}{32} \Box^{2} \ln \mu^{2}x^{2}$$

etc., where $\Box = \partial_{i}\partial_{i}$ is the usual Laplacian, $x^{4} = (x^{2})^{2}$, $x^{6} = (x^{2})^{3}$, and $\mu, \mu'$ are massive parameters which play the role of subtraction points. For $x \neq 0$, the expressions in the left-hand side and the right-hand side of (3–5) are identical. By definition, the extension of functionals in the left-hand side from the subspace of test functions which vanish near $x = 0$ to the whole space is determined by the right-hand side. Note that all the derivatives involved are
understood in the distributional sense, i.e. a derivative $D^af$ of a distribution $f$ acts on a test function $\phi$ as

$$ (D^af, \phi) = (-1)^{|x|}(f, D^a\phi), $$

where $|x|$ being the order of the derivative.

In [7] a second version of differential renormalization was presented in the case of scalar massless logarithmically divergent diagrams. It was based on 'pulling out' another differential operator instead of the Laplacian. In particular, (3) is replaced by

$$ \frac{1}{x^2} \to S \ln \mu^2 x^2 x^4, $$

where

$$ \hat{S} = \frac{1}{2} \frac{\partial}{\partial x_a} x_a. $$

Within this version, there is no necessity of reducing the problem of renormalization to propagator-type diagrams. (This reduction is as usual important in renormalization group calculations–see below.) Thus there is no asymmetry of treating vertices of the given graph.

The purpose of this paper is to present a general prescription of this version of differential renormalization which is applicable for arbitrary diagrams including massive ones. The status of the renormalized action principle within differential renormalization will be also discussed. Another task is to show that some constants that naturally arise within this approach [7] are straightforwardly related to the renormalization group coefficients. It will be proved that the beta function and anomalous dimensions are expressed through these constants by the same formulae that, in the case of the MS scheme, the RG coefficients are expressed through counterterms. Note that the differential renormalization happens to be a mass-independent scheme.

The plan of the paper is as follows. In the next section necessary differential operators similar to (8) will be presented and standard formulae for the $R$-operation will be listed. Then in Sect. 3 renormalization of massless lower order diagrams is characterized. In Sect. 4 an auxiliary technique necessary for renormalization in the massive case is introduced through examples of lower order graphs. In Sect. 5 the general prescriptions are formulated and justified. Sect. 6 is devoted to discussion of the action principle within differential renormalization. In Sect. 7 explicit formulae for RG coefficients will be obtained. Finally, Sect. 8 contains discussion of the results obtained.

### 2 Notation

#### 2.1 Differential operators

Let us define the following differential operator:

$$ \hat{S}_x = \frac{1}{2} \sum_{i=1}^{n} \frac{\partial}{\partial x_{ia}} (x_{ia} - \bar{x}_a), $$

where $\bar{x} = x_1, \ldots, x_n$ is a set of $n$ four-dimensional variables, and $x = (1/n) \sum x_i$. If $F(x)$ is a translationally invariant function, i.e. $F(x + a) = F(x)$, then $F(x) = f(y)$ for $u_i = x_i - x_{i_0}, i \neq i_0$ and

$$ \hat{S}_x F(x) = \hat{S}_n f(y), $$

where

$$ \hat{S}_n = \frac{1}{2} \sum_{i < j} \frac{\partial}{\partial u_{ij}} u_{ij}. $$

Since the Feynman amplitudes are translationally invariant we will use subsequently this form for the operator $\hat{S}$.

In fact, the homogeneity properties of Feynman amplitudes play an essential role. If $f(u_1, \ldots, u_{n-1})$ is a homogeneous function or distribution of degree $\lambda$ then

$$ \hat{S}f = \frac{1}{2} (\lambda + 4(n - 1)) f. $$

Note that the operator $\hat{S}$ involves a preliminary multiplication by variables $u_i$ which vanish at points where initial amplitudes are singular. Correspondingly, these singularities are reduced. When the ultraviolet divergence is logarithmic it then disappears if the subsequent differentiation is understood in the distributional sense–see (6).

In the case of linear divergences multiplication by a monomial of the first degree in coordinates is not sufficient. A second order monomial is necessary so that it is natural to apply the following operator

$$ \hat{S}^{(1)} = \frac{1}{4} \sum_{i,j,a,b} \frac{\partial}{\partial x_{ia}} \frac{\partial}{\partial x_{jb}} (x_{ia} - \bar{x}_a) (x_{jb} - \bar{x}_b). $$

For massless graphs, it is sufficient to apply (11), (12) and their generalizations. If massive lines are present, we may use homogeneity of Feynman amplitudes with respect to coordinates and inverse masses. Then a natural analog of (11) is given by

$$ \hat{S} = \frac{1}{2} \sum_i \frac{\partial}{\partial u_i} u_i - \frac{1}{2} \sum_i \frac{\partial}{\partial m_i} m_i, $$

since differentiation in masses also improves the ultraviolet behaviour.

In the general case of degree of divergence $\omega$ let us apply the following differential operator:

$$ \hat{S}^{(\omega)} = N \{ \hat{S}^0 \ldots \hat{S}^0 \}, $$

where $\omega$ is the degree of divergence, $\hat{S}^0$ is defined by (13), and the symbol of the $N$-product implies that all the derivatives $\partial/\partial u_i$ are to the left of all $u_i$, while all the derivatives $\partial/\partial m_j$ are to the right of $m_j$. It is not difficult to show that

$$ \hat{S}^{(\omega)} = \hat{S}(\hat{S} + 1/2) \ldots (\hat{S} + \omega/2). $$