ON THE ALGEBRAIC STRUCTURE OF PARTICLE MOTION
IN A FIELD OF TURBULENCE

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Summary: The motion of particles in a turbulent flow is described by means of algebraic physics. The initial concepts are structurally ordered groupoids, algebras of observables, logically dependent on them, with couplings and the non-canonic transition between two Hamiltonians. The non-canonic transition leads to the substitution of the time t by a new parameter. Its real counterpart fixes the lower limit of the size of the time step in the differential equation of transfer, based on the semi-empirical image of turbulent diffusion.

1. INTRODUCTION

We are witnesses to the increasingly important role of algebraic description as a significant part of current mathematical modelling and solving of problems of mathematical physics. Algebraic description, expressing by methods of general algebra the relations in the sequence of causes and consequences, which relate the appropriate quantities, has a large and frequently also decisive influence on the development of many branches of physics, including the hydrodynamics of the ideal fluid. New relations have been found between fluid flow and Lie groups, and the problems of hydrodynamic instability have been formulated as problems of geodesics of left-invariant metrics on Lie groups [1–3].

Equally important is the connection of group-algebra methods with the invariant (frequently new) properties of physical systems under non-canonic transition between two Hamiltonians [4].

Our task will be to determine what results we can achieve by applying algebraic methods, based on the concept of a universal algebra of a special type, referred to as the structuroid in [5], in the problem of particle motion in a turbulent flow. In [6], the non-canonic transformation, included in the description of particle transfer in a turbulent field, had already become part of the algebraic approach to the phenomenon being considered. However, only in relation to the problem of the minimum algebra of observables in the kinetic model of turbulent diffusion.

In this case, we are interested in a more general approach to particle motion in a field of turbulence as provided by the combination of the field theory with the formalism of the non-canonic transition between two Hamiltonians. We assume that Hamilton’s principle can also be used in this case, although the only criterion that it is valid can no longer be provided by the agreement of the equation of motion, resulting from this principle, with the equation derived in a different way.

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Let $G$ be an algebra above body $Q$ with the set of operations of binary multiplication $\circ$, and $s_1, s_2$ linear subspaces sleeved over elements $a_1, a_2, \ldots, a_k$ of the base of algebra $G$ ($G$ is simultaneously the linear vector field above $Q$).

We shall refer to the linear subspace, sleeved over all possible products $a_{k_1}^{(1)} \circ a_{k_2}^{(2)}$ as the product of linear subspaces $s_1 \circ s_2$ sleeved over $a_{k_1}^{(1)}$ and $a_{k_2}^{(2)}$ of linear subspaces $s_1$ and $s_2$. Set $S_G$ of all possible linear subspaces $s$ with respect to the multiplication operation $\circ$ forms a groupoid and, with a view to the operations of union $\cup$ and intersection $\cap$, a structure. It holds that $(s_1 \cup s_2) \circ s_3 = (s_1 \circ s_3) \cup (s_2 \circ s_3)$, $(s_1 \cap s_2) \circ o s_3 \subseteq (s_1 \circ s_3) \cap (s_2 \circ s_3)$. Inclusion $\subseteq$ also occurs in the last relation next to the equality since, after multiplying by $s_3$ in the non-intersecting parts of subspaces $s_1$ and $s_2$, common elements may appear. The universal algebra whose elements are the linear subspaces from $G$ with the algebraic operations $\cup, \cap$ and $\circ$, is the structoroid $S_G$ over algebra $G$. $S_G$ is the realization of the abstract structoroid whose elements are subject to the same algebraic relations, but without the possibility of envisaging them as linear subspaces.*

Element $s \in S_G$ is called the subalgebra of algebra $G$ if $s \circ s \subseteq s$, and subalgebra $s$ is the left (or right) ideal and, at the same time, $G \circ s \subseteq s$ (or $s \circ G \subseteq s$), respectively.

Algebras with two binary operations are of principal importance in the algebraic theory of physical systems, i.e. the Lie-Jordan algebras of observables and the Lie associative algebras they determine. It is evident that Lie-Jordan and Lie associative structuroids will correspond to them. It holds that $[s_1, (s_2, s_3)] = (s_1 \circ s_3) \cup (s_2 \circ s_3)$, $[s_1, s_2, s_3] \subseteq (s_1 \circ s_3) \cup (s_2 \circ s_3)$.

Now, let $F$ be a certain linear subspace of $G$. By the structural form of the relation for $F$ from $G$ we shall then understand the relation of the type $[F, F] = F$, where $[, ,]$ is the Lie product, supplemented by the condition $GF = FG \subseteq F$. It evidently follows that subspace $F$ is simultaneously the Lie algebra $(GF = FG)$ and the associative ideal $(FG \subseteq F)$ in algebra $G$.

In connection with the concept of the structoroid, the algebra of observables with bonds (this is indeed the set of all bonds which is the linear subspace $F$ of algebra $G$) can be described generally, if we take into account the concept of algebraic state ("\(\pi\)-state" for short). It is understood to be the logical consequence of the general need to describe physical systems and to solve dynamic problems by algebraic methods. It is necessary to establish a relation between the algebras of observables and the real numerical values, determined experimentally, which we identified with the mean values of the observables.

*) The definition of structoroid $S_G$ over $G$ can be generalized to the case of universal algebras of general form [5] on which certain sets $\Omega$ of $n$-dimensional algebraic operations are defined. The universal algebra $S_G$ will then be a structoroid over universal algebra $G$ with the set of operations.

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