Instantons and Algebraic Geometry

M. F. Atiyah and R. S. Ward
Mathematical Institute, University of Oxford, Oxford OX1 3LB, England

Abstract. Minimum action solutions for SU(2) Yang-Mills fields in Euclidean 4-space correspond, via the Penrose twistor transform, to algebraic bundles on the complex projective 3-space. These bundles in turn correspond to algebraic curves. The implication of these results for the Yang-Mills fields is described. In particular all solutions are rational and can be constructed from a series of Ansätze $A_l$ for $l \geq 1$.

§ 1. Introduction

The term instanton or pseudo-particle has been coined for the minimum action solutions of SU(2) Yang-Mills fields in Euclidean 4-space $\mathbb{R}^4$. Conditions at infinity are imposed which are tantamount to working on the 4-sphere $S^4$ (which is the conformal compactification of $\mathbb{R}^4$) and are classified by an integer $k$, which is interpreted as the "number of instantons". The most general solutions so far constructed explicitly are those of Jackiw et al. [4]. They depend on $5|k| + 4$ real parameters if $|k| \geq 3$, while for $|k| = 1, 2$ the number of parameters are 5, 13 respectively. On the other hand infinitesimal deformation theory shows that the number of parameters for the complete family of solutions is $8|k| - 3$ (see [1, 5, 9]). The purpose of this note is to describe how the full $(8|k| - 3)$-parameter family can in principle be constructed by using algebraic geometry.

The basic idea is to use the Penrose Twistor approach to space-time [8] in which field equations in 4-space are converted into complex analytic geometry on complex projective 3-space $P_3$. This approach was applied in [11] to the self-dual (or anti-self dual) Yang-Mills equations (which correspond to minimum action). The resulting geometrical objects on $P_3$ turn out to be complex analytic bundles. This transformation can be applied both locally and globally and for either Minkowski or Euclidean space. For the instanton problem we take the global Euclidean version (i.e. for $S^4$) and this leads to a complex analytic bundle defined over the whole of $P_3$. By Serre's basic theorems of analytic geometry [10] such bundles are necessarily algebraic. Moreover algebraic geometers have recently made significant progress on the study of precisely those algebraic bundles which correspond to the SU(2) Yang-
Mills equations [2, 3, 6]. We shall apply these results and interpret them back in the Euclidean picture.

From a computational point of view the conclusion may be described briefly as follows. The instanton solutions so far known have been constructed by an Ansatz due to t'Hooft. This Ansatz which we shall label $A_1$ starts from a suitable solution of the (linear) Laplace equation in 4-space. From the algebraic geometry we find that there is a whole hierarchy of Ansätze $A_l$ ($l=1, 2, \ldots$). The Ansatz $A_2$ starts from a solution of the (linear) anti-self-dual Maxwell equations while $A_1$ for $l \geq 3$ uses the corresponding first order equations for fields of spin $(l-1)$. The solutions obtained from $A_l$ are included in those coming from $A_{l+1}$ but not vice-versa in general. For $k=1, 2$ the Ansatz $A_1$ suffices to give $(8k-3)$-parameters but for $k=3, 4$ we must use $A_2$. For any given $k$ there exists a large integer $l(k)$ so that all solutions come from $A_{l(k)}$. There are conjectural values of $l(k)$ but these are not yet established.

For solutions to be globally well-defined on $S^4$ the solutions of the linear equations have to be suitably chosen. For $A_1$ the appropriate solution $\phi$ of the Laplace equation $\Delta \phi = 0$ has $\frac{1}{r^2}$ singularities at a given set of points. For $A_2$ the appropriate solution of the anti-self-dual Maxwell equations has a singularity along a 2-dimensional surface in $\mathbb{R}^4$. However this surface is not arbitrary: it corresponds to an elliptic curve in $P_3$. Topologically it is a torus (with some self-crossing, i.e. double points) but analytically it is constrained to satisfy a certain differential equation which can be described in terms of its second fundamental form.

The algebraic character of our solutions is reflected in the fact that the field $F$ on 4-space is, in a suitable gauge, given by rational functions. In particular the action density is rational and its poles in the complex domain play an important role.

In this note detailed mathematical arguments will not be given. Instead we concentrate on describing the conclusions. A fuller treatment which necessarily involves extensive use of modern algebraic geometry will appear elsewhere.

Acknowledgment. We are much indebted to I.M.Singer for interesting us in Yang-Mills fields and to R.Penrose, N.J.Hitchin, and R.Hartshorne for many stimulating and useful discussions.

§ 2. The Penrose Transform

In this section we shall describe briefly how the Penrose theory leads to a reinterpretation of the self-dual Yang-Mills equations. Penrose's starting point is the observation that the complexification $Q_4$ of $S^4$ given by the homogeneous equation

$$\sum_{i=1}^{5} z_i^2 = z_6^2$$

in $P_5$ can be identified with the Grassmannian of lines in $P_3$. This goes back to Felix Klein and is based on the Plücker coordinates $\pi_{ij} = x_i y_j - x_j y_i$ of the line $x, y$ in $P_3$. The $\pi_{ij}$ are skew-symmetric, determined up to a scalar and satisfy the quadratic identity

$$\pi_{01} \pi_{23} + \pi_{02} \pi_{31} + \pi_{03} \pi_{12} = 0$$

which by a suitable complex change of coordinates is identified with the equation

$$\sum_{i=1}^{5} z_i^2 = z_6^2$$

of $Q_4$. One now transforms problems on $S^4$ by first complexifying them to $Q_4$ and then using the Klein correspondence to pass to $P_3$. Applying this Penrose